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Journal of Geometry and Physics 58 (2008) 701–719

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Induction for weak symplectic Banach manifolds

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Received 24 December 2007; accepted 22 January 2008
Available online 1 February 2008

Abstract

The symplectic induction procedure is extended to the case of weak symplectic Banach manifolds. Using this procedure, one constructs hierarchies of integrable Hamiltonian systems related to the Banach Lie–Poisson spaces of k-diagonal trace class operators.

MSC: 53D05; 53D17; 53Z05; 37J35; 46N20; 46T05

Keywords: Banach Lie–Poisson space; Momentum map; Semi-infinite systems; Coadjoint orbit; Action-angle variables

1. Introduction

The theory of symplectic and Poisson Banach manifolds provides a solid mathematical foundation for the investigation of infinite dimensional Hamiltonian systems that appear in various domains of mathematics and physics. For finite dimensional Hamiltonian systems one uses mainly differential geometric methods. In order to study infinite dimensional systems it is necessary to appeal to functional analytic methods, which renders their study more difficult. For example, the theory of Banach Lie–Poisson spaces, which extends Hamiltonian mechanics on duals of Lie algebras to the Banach space context, is intimately related to the theory of W^*-algebras (see [7]).

On weak symplectic manifolds not all smooth functions admit a Hamiltonian vector field since the map from the tangent space to the cotangent space of the manifold induced by the weak symplectic form is only injective. We introduce in Section 2 the Poisson subalgebra of smooth functions that admit Hamiltonian vector fields and use it to define momentum maps of Lie algebra actions on weak symplectic manifolds. It turns out that the crucial finite dimensional property of equivariant momentum maps being Poisson holds here too when making the following modifications: the space of functions on the weak symplectic manifold consists only of those that admit Hamiltonian vector fields and the dual of the Lie algebra of symmetries is replaced by the Banach Lie–Poisson space given by the predual of this Lie algebra.

The theory of symplectic induction on weak symplectic manifolds is presented in Section 3. Explicit formulas for the induced weak symplectic form and the momentum map (in the sense described above) are given.

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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2008.01.003

The rest of the paper is devoted to the study of an example. In Section 4 several families of trace class operators on a real separable Hilbert space are introduced. A prominent role is played by the k-diagonal trace class operators, the corresponding Banach Lie algebra (obtained as the dual of this space), and the underlying Banach Lie groups. Many explicit formulas are presented and several results from [8] needed for the next section are recalled here.

In Section 5 the symplectic induction procedure is applied to the concrete example of the weak symplectic manifold $(\ell^\infty \times \ell^1, \omega)$, the Banach Lie group of k-diagonal operators, and the Banach Lie subgroup of the k-bidiagonal operators, both introduced and studied in detail in Section 4. The end result of the induction procedure is the weak symplectic Banach manifold $\left((\ell^\infty)^{k-1} \times (\ell^1)^{k-1}, \Omega_k\right)$, where the weak symplectic form Ω_k turns out not to be the canonical one. The associated momentum map given by induction is also presented. It turns out that the momentum map obtained by the induction method is a generalization of the classical Flaschka map appearing in the theory of the Toda lattice (see [4]). Indeed, if $k = 2$ the symplectic induced space coincides with $(\ell^\infty \times \ell^1, \omega)$ and the associated momentum map is identical to the Flaschka map for the semi-infinite Toda lattice. However, the general case presented here constructs other hierarchies of integrals in involution for other systems generalizing the Toda lattice. All formulas are worked out in detail for the case $k = 3$.

Conventions. In this paper all Banach manifolds and Lie groups are real. The definition of the notion of a Banach Lie subgroup follows Bourbaki [2], that is, a subgroup H of a Banach Lie group G is necessarily a submanifold (and not just injectively immersed). In particular, Banach Lie subgroups are necessarily closed.

2. Momentum map on a weak symplectic manifold

The goal of this section is to present some notions which are indispensable for the procedure of symplectic induction on weak symplectic manifolds. In the process we shall define the concepts of weak symplectic manifold, the associated Poisson algebra, Banach Lie–Poisson space, and the momentum map. We shall also establish some of their elementary properties and give examples relevant to the subsequent developments in this paper.

Weak symplectic manifolds. In infinite dimensions there are two possible generalizations of the notion of a symplectic manifold.

Definition 2.1. Let P be a Banach manifold and ω a two-form. Then ω is said to be *weakly nondegenerate* if for every $p \in P$ the map $v_p \in T_pP \mapsto \omega(p)(v_p, \cdot) \in T_p^*P$ is injective. If, in addition, this map is also surjective, then the form ω is called *strongly nondegenerate*. The form ω is called a *weak* or *strong symplectic form* if, in addition, $\mathbf{d}\omega = 0$, where $\mathbf{d}$ denotes the exterior differential on forms. The pair (P, ω) is called a *weak* or *strong symplectic manifold*, respectively.

If P is finite dimensional this distinction does not occur since every linear injective map of a vector space into itself is also surjective. The typical example of an infinite dimensional strongly symplectic Banach manifold is a complex Hilbert space endowed with the symplectic form equal to the imaginary part of the Hermitian inner product. Any strong symplectic form is locally constant, but weak symplectic forms are not in general (see e.g. Section 3.2 in [1]). The usual Hamiltonian formalism extends from the finite dimensional setting to the strong symplectic case without any difficulties.

The first problem that arises when working on a weak symplectic Banach manifold (P, ω) is that one cannot define the associated Poisson bracket $\{f, g\}_\omega$ for arbitrary $f, g \in C^\infty(P)$. The reason is that the linear continuous map $T_pP \ni v_p \mapsto \flat_p(v_p) := \omega(p)(v_p, \cdot) \in T_p^*P$ is only injective. Then, in order to define the Hamiltonian vector field X_f by the usual formula

$$\mathbf{i}_{X_f}\omega = \mathbf{d}f \tag{2.1}$$

one needs the condition that $\mathbf{d}f(p) \in \flat_p(T_pP)$ for all $p \in P$. Let us denote by $C^\infty_\omega(P) \subset C^\infty(P)$ the vector space of smooth functions which satisfy the above condition. If $f, g \in C^\infty_\omega(P)$, the Hamiltonian vector fields X_f and X_g exist and one defines the Poisson bracket, as usual, by

$$\{f, h\}_\omega := \omega(X_f, X_h) = \langle \mathbf{d}f, X_h\rangle = \boldsymbol{\pounds}_{X_h} f. \tag{2.2}$$

Since $X_{fg} = fX_g + gX_f$ whenever $f, g \in C^\infty_\omega(P)$ it follows that $C^\infty_\omega(P)$ is an algebra relative to the pointwise multiplication of functions and that the Leibniz identity holds.

Using the identities

$$\mathbf{i}_{[X_f,X_g]}\omega = \left(\pounds_{X_f} \circ \mathbf{i}_{X_g} - \mathbf{i}_{X_g} \circ \pounds_{X_f}\right)\omega$$

and $\pounds_{X_f}\omega = \mathbf{d}\mathbf{i}_{X_f}\omega = \mathbf{d}^2 f = 0$ for any $f, g \in C^\infty_\omega(P)$, one obtains

$$\mathbf{i}_{[X_f,X_g]}\omega = \mathbf{d}X_f[g] = \mathbf{d}\omega(X_g, X_f) \tag{2.3}$$

which shows that $C^\infty_\omega(P)$ is closed with respect to the Poisson bracket $\{\,,\}_\omega$. In addition (2.2) and (2.3) show that

$$[X_f, X_g] = -X_{\{f,g\}_\omega} \tag{2.4}$$

which is equivalent to the Jacobi identity in $C^\infty_\omega(P)$. Summarizing, we have proved the following.

Proposition 2.2. *The algebra $C^\infty_\omega(P)$ is a Poisson algebra, that is, it is an algebra relative to multiplication of functions, it is a Lie algebra relative to the Poisson bracket $\{\,,\}_\omega$, and the Leibniz identity holds.*

So, as opposed to the strong symplectic case, the Poisson algebra $C^\infty_\omega(P)$ defined by the weak symplectic form ω is smaller than $C^\infty(P)$. But $C^\infty(P)$ is a $C^\infty_\omega(P)$-module because any $f \in C^\infty_\omega(P)$ acts on $g \in C^\infty(P)$ by $g \mapsto X_f[g]$. In addition, the subalgebra $C^\infty_\omega(P)$ is invariant with respect to this action.

Substituting X instead of X_f in (2.3), where X is a locally Hamiltonian vector field, that is, it satisfies $\pounds_X\omega = 0$, we see that $X[g] \in C^\infty_\omega(P)$ (in fact, the Hamiltonian vector field of $X[g]$ is $[X, X_g]$). Thus, $C^\infty_\omega(P)$ is invariant with respect to the action of the Lie algebra of locally Hamiltonian vector fields.

Finally, note that the Poisson bracket $\{f, g\}_\omega(p)$ for $f, g \in C^\infty_\omega(P)$ is completely determined by $\mathbf{d}f(p)$ and $\mathbf{d}g(p)$.

The weak symplectic manifold $(\ell^\infty \times \ell^1, \omega)$. Now let us present as a canonical example the weak symplectic manifold $(\ell^\infty \times \ell^1, \omega)$. The space $\ell^\infty \times \ell^1$ is the Banach space product of the Banach space ℓ^∞ of bounded real sequences and the Banach space ℓ^1 of absolutely convergent sequences, that is,

$$\mathbf{q} := \{q_k\}_{k=0}^\infty \in \ell^\infty \quad \text{if and only if} \quad \|\mathbf{q}\|_\infty := \sup_{k=0,1,\ldots} |q_k| < +\infty$$

and

$$\mathbf{p} := \{p_k\}_{k=0}^\infty \in \ell^1 \quad \text{if and only if} \quad \|\mathbf{p}\|_1 := \sum_{k=0}^\infty |p_k| < +\infty.$$

The strongly nondegenerate duality pairing

$$\langle \mathbf{q}, \mathbf{p}\rangle = \sum_{k=0}^\infty q_k p_k, \quad \text{for } \mathbf{q} \in \ell^\infty, \mathbf{p} \in \ell^1, \tag{2.5}$$

establishes the Banach space isomorphism $(\ell^1)^* = \ell^\infty$. The weak symplectic form ω is the canonical one given by

$$\omega((\mathbf{q}, \mathbf{p}), (\mathbf{q}', \mathbf{p}')) = \langle \mathbf{q}, \mathbf{p}'\rangle - \langle \mathbf{q}', \mathbf{p}\rangle, \quad \text{for } \mathbf{q}, \mathbf{q}' \in \ell^\infty, \mathbf{p}, \mathbf{p}' \in \ell^1. \tag{2.6}$$

for $\mathbf{q}, \mathbf{q}' \in \ell^\infty$ and $\mathbf{p}, \mathbf{p}' \in \ell^1$. It is useful to express ω, like in finite dimensions, as

$$\omega = \sum_{k=0}^\infty \mathbf{d}q_k \wedge \mathbf{d}p_k \tag{2.7}$$

relative to the coordinates q_k, p_k. The expression (2.7) needs the following functional analytic interpretation. The standard Schauder basis $\{|k\rangle\}_{k=0}^\infty$ of ℓ^1 induces the basis $\{\partial/\partial p_k\}_{k=0}^\infty$ on the tangent space $T_{\mathbf{p}}\ell^1$, which, of course, coincides with ℓ^1. On ℓ^∞ the same basis is interpreted as follows. Any $\mathbf{a} := \{a_k\}_{k=0}^\infty \in \ell^\infty$ can be uniquely written as a *weakly* convergent series $\mathbf{a} = \sum_{k=0}^\infty a_k|k\rangle$ and hence for $\mathbf{q} \in \ell^\infty$ we define the sequence $\{\partial/\partial q_k\}_{k=0}^\infty$ of elements in the tangent space $T_{\mathbf{q}}\ell^\infty \cong \ell^\infty$ to correspond to $\{|k\rangle\}_{k=0}^\infty$.

With this understanding of the basis $\{\partial/\partial q_k, \partial/\partial p_k\}_{k=0}^{\infty}$ of $T_{(\mathbf{q},\mathbf{p})}(\ell^\infty \times \ell^1)$ we can write any smooth vector field $X \in \mathfrak{X}(\ell^\infty \times \ell^1)$ as

$$X(\mathbf{q}, \mathbf{p}) = \sum_{k=0}^{\infty} \left(A_k(\mathbf{q}, \mathbf{p}) \frac{\partial}{\partial q_k} + B_k(\mathbf{q}, \mathbf{p}) \frac{\partial}{\partial p_k} \right),$$

where $\{A_k(\mathbf{q}, \mathbf{p})\}_{k=0}^{\infty} \in \ell^\infty$ and $\{B_k(\mathbf{q}, \mathbf{p})\}_{k=0}^{\infty} \in \ell^1$. Thus, if Y is another vector field whose coefficients are $\{C_k(\mathbf{q}, \mathbf{p})\}_{k=0}^{\infty} \in \ell^\infty$, $\{D_k(\mathbf{q}, \mathbf{p})\}_{k=0}^{\infty} \in \ell^1$, applying formally the exterior differential calculus suggested by formula (2.7) we get

$$\left(\sum_{k=0}^{\infty} \mathbf{d}q_k \wedge \mathbf{d}p_k \right) (X, Y)(\mathbf{q}, \mathbf{p}) = \sum_{k=0}^{\infty} (A_k(\mathbf{q}, \mathbf{p}) D_k(\mathbf{q}, \mathbf{p}) - C_k(\mathbf{q}, \mathbf{p}) B_k(\mathbf{q}, \mathbf{p}))$$

which coincides with (2.6). It is in this sense that formula (2.7) represents the weak symplectic form (2.6).

In this case we can determine explicitly the space $C^\infty_\omega(\ell^\infty \times \ell^1)$. To do this, we observe that for any $h \in C^\infty_\omega(\ell^\infty \times \ell^1)$ its partial derivatives $\partial h/\partial \mathbf{q} \in (\ell^\infty)^*$ and $\partial h/\partial \mathbf{p} \in (\ell^1)^* = \ell^\infty$, respectively. Since $\flat_{(\mathbf{q},\mathbf{p})}(\ell^\infty \times \ell^1) \cong \ell^1 \times \ell^\infty \subset (\ell^\infty)^* \times (\ell^1)^*$ one concludes that the Hamiltonian vector field X_h defined by the weak symplectic form (2.7) and the function h exists if and only if $\partial h/\partial \mathbf{q} \in \ell^1 \subset (\ell^1)^{**} = (\ell^\infty)^*$. Therefore,

$$C^\infty_\omega(\ell^\infty \times \ell^1) = \{f \in C^\infty(\ell^\infty \times \ell^1) \mid \{\partial h/\partial q_k\}_{k=0}^{\infty} \in \ell^1\}, \tag{2.8}$$

and the Hamiltonian vector field defined by $h \in C^\infty_\omega(\ell^\infty \times \ell^1)$ has the expression

$$X_h(\mathbf{q}, \mathbf{p}) = \frac{\partial h}{\partial p_k} \frac{\partial}{\partial q_k} - \frac{\partial h}{\partial q_k} \frac{\partial}{\partial p_k}. \tag{2.9}$$

The canonical Poisson bracket of $f, h \in C^\infty_\omega(\ell^\infty \times \ell^1)$ makes sense and is given by

$$\{f, g\}_\omega(\mathbf{q}, \mathbf{p}) = \sum_{k=0}^{\infty} \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k} \frac{\partial f}{\partial p_k} \right). \tag{2.10}$$

Banach Lie–Poisson spaces. We recall from [7] the following facts. Consider a Banach Lie algebra $(\mathfrak{g}, [\,,\,])$, that is, $\mathfrak{g}$ is a Banach space and the Lie bracket operation $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is continuous in the norm topology. By definition, a *Banach Lie–Poisson space* is a Banach space $\mathfrak{g}_*$ predual to $\mathfrak{g}$, that is, $(\mathfrak{g}_*)^* = \mathfrak{g}$, such that $\mathrm{ad}_x^* \mathfrak{g}_* \subset \mathfrak{g}_*$ for all $x \in \mathfrak{g}$. Recall that $\mathrm{ad}_x^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual map to $\mathrm{ad}_x := [x, \cdot] : \mathfrak{g} \to \mathfrak{g}$ and that $\mathfrak{g}_*$ is a Banach subspace of $\mathfrak{g}^*$. Under these assumptions one defines the Poisson bracket of $f, h \in C^\infty(\mathfrak{g}_*)$ by

$$\{f, h\}(\rho) = \langle [Df(\rho), Dh(\rho)], \rho \rangle, \tag{2.11}$$

where $\rho \in \mathfrak{g}_*$ and $Df(\rho) \in \mathfrak{g}$ denotes the Fréchet derivative of f at the point ρ. The bracket equation (2.11) makes $C^\infty(\mathfrak{g}_*)$ into a Poisson algebra admitting $(\mathfrak{g}_*)^* = \mathfrak{g} \subset C^\infty(\mathfrak{g}_*)$ as a subalgebra. Moreover, the Poisson bracket equation (2.11) coincides with the original Lie bracket $[\,,\,]$ on $\mathfrak{g}$.

The crucial example of a Banach Lie–Poisson space is the Banach space L^1 of trace class operators on a real or complex separable Hilbert space $\mathcal{H}$. It is well known that the Banach space $(L^1)^*$ dual to L^1 is canonically isomorphic to the Banach Lie algebra $(L^\infty, [\,,\,])$ of bounded linear operators with the commutator as the Lie bracket. The isomorphism $(L^1)^* \cong L^\infty$ is given by

$$\langle x, \rho \rangle = \mathrm{Tr}(\rho x), \tag{2.12}$$

where $x \in L^\infty$ and $\rho \in L^1$ (see, e.g. [9]).

Momentum maps on weak symplectic manifolds. The finite dimensional definition of the momentum map (see, e.g., [1] or [6]) generalizes to weak symplectic Banach manifolds or Banach Poisson manifolds if one imposes certain conditions specific to the infinite dimensional setting. We begin with a direct extension of the finite dimensional setting.

Assume that $\mathfrak{g}$ is a Lie algebra whose predual $\mathfrak{g}_*$ is a Banach Lie–Poisson space.

Definition 2.3. Let (P, ω) be a weak symplectic Banach manifold and assume that $\mathfrak{g}$ acts smoothly on P on the left, that is, there is a smooth map $(x, p) \in \mathfrak{g} \times P \mapsto x_P(p) \in TP$ such that $[x, y]_P = -[x_P, y_P]$ for any $x, y \in \mathfrak{g}$, that is, one has a Lie algebra anti-homomorphism of $\mathfrak{g}$ into the Lie algebra $\mathfrak{X}(P)$ of smooth vector fields on P. A *momentum map* $\mathbf{J} : P \to \mathfrak{g}_*$ is defined by the conditions

(i) $x \circ \mathbf{J} \in C^\infty_\omega(P)$ for all $x \in \mathfrak{g}$, and
(ii) $\mathbf{i}_{x_P}\omega = \mathbf{d}(x \circ \mathbf{J})$, that is, $x_P = X_{x\circ\mathbf{J}}$, for all $x \in \mathfrak{g}$.

If, in addition, $T_p\mathbf{J}\,(x_P(p)) = -\operatorname{ad}^*_x \mathbf{J}(p)$ for any $x \in \mathfrak{g}$, $p \in P$, then the momentum map $\mathbf{J}$ is said to be *infinitesimally equivariant*.

In the Banach Poisson manifold context, it is convenient to introduce momentum maps differently (see [7]).

Definition 2.4. A smooth map $\mathbf{J} : P \to \mathfrak{g}_*$ that satisfies

(i) $x \circ \mathbf{J} \in C^\infty_\omega(P)$ for all $x \in \mathfrak{g}$, and
(ii) $\{x \circ \mathbf{J}, y \circ \mathbf{J}\}_\omega = [x, y] \circ \mathbf{J}$ for all $x, y \in \mathfrak{g}$

is called a *momentum map*.

As we shall see below, this definition automatically implies infinitesimal equivariance relative to linear functions on the predual $\mathfrak{g}_*$. The relationship between these definitions is given by the following statement.

Proposition 2.5. *An infinitesimally equivariant momentum map in the sense of Definition 2.3 is a momentum map in the sense of Definition 2.4. Conversely, given a momentum map in the sense of Definition 2.4, there is a smooth Lie algebra action that admits a momentum map in the sense of Definition 2.3 which is infinitesimally equivariant.*

Proof. Assume that $\mathbf{J} : P \to \mathfrak{g}_*$ is an infinitesimally equivariant momentum map in the sense of Definition 2.3. Then applying the infinitesimal equivariance condition to $y \in \mathfrak{g}$ and using the conditions (i) and (ii) in Definition 2.3, we get

$$\begin{aligned}([x, y] \circ \mathbf{J})\,(p) &= \langle \operatorname{ad}_x y, \mathbf{J}(p)\rangle = \left\langle y, \operatorname{ad}^*_x \mathbf{J}(p)\right\rangle = -\left\langle y, T_p\mathbf{J}(x_P(p))\right\rangle = -\mathbf{d}(y \circ \mathbf{J})(p)(x_P(p)) \\ &= -\mathbf{d}(y \circ \mathbf{J})(p)(X_{x\circ\mathbf{J}}(p)) = \{x \circ \mathbf{J}, y \circ \mathbf{J}\}_\omega(p)\end{aligned}$$

by (2.2). Therefore, $\mathbf{J}$ is a momentum map in the sense of Definition 2.4.

Conversely, assume that $\mathbf{J}$ is an infinitesimally equivariant momentum map in the sense of Definition 2.4. The smooth map $(x, p) \in \mathfrak{g} \times P \mapsto X_{x\circ\mathbf{J}}(p) \in TP$ defines indeed a Lie algebra action of $\mathfrak{g}$ on P by setting $x_P(p) := X_{x\circ\mathbf{J}}(p)$. Indeed, by (2.4) and (ii) in Definition 2.4 we get

$$[x_P, y_P] = [X_{x\circ\mathbf{J}}, X_{y\circ\mathbf{J}}] = -X_{\{x\circ\mathbf{J}, y\circ\mathbf{J}\}_\omega} = -X_{[x,y]\circ\mathbf{J}} = -[x, y]_P$$

as required. Condition (ii) in Definition 2.3 is satisfied by the construction of the action and conditions (i) in both definitions coincide. It remains to show that $\mathbf{J}$ so defined is indeed infinitesimally equivariant. Indeed, by (ii) of Definition 2.4, for any $x, y \in \mathfrak{g}$ we have

$$\begin{aligned}\left\langle y, T_p\mathbf{J}(x_P(p))\right\rangle &= \mathbf{d}(y \circ \mathbf{J})(p)(x_P(p)) = \mathbf{d}(y \circ \mathbf{J})(p)(X_{x\circ\mathbf{J}}(p)) = -\{x \circ \mathbf{J}, y \circ \mathbf{J}\}_\omega(p) \\ &= -\left([x, y] \circ \mathbf{J}\right)(p) = -\langle \operatorname{ad}_x y, \mathbf{J}(p)\rangle = -\left\langle y, \operatorname{ad}^*_x \mathbf{J}(p)\right\rangle\end{aligned}$$

so that by the Hahn–Banach Theorem we conclude that $T_p\mathbf{J}(x_P(p)) = -\operatorname{ad}^*_x \mathbf{J}(p)$ for any $x \in \mathfrak{g}$ and any $p \in P$. ■

Remarks A. Condition (ii) in Definition 2.4 can be interpreted as expressing the fact that $\mathbf{J}$ is a Poisson map relative to continuous linear functions on $\mathfrak{g}_*$, thought of as a Banach Lie–Poisson space. Therefore, using the Leibniz identity, $\mathbf{J}$ is Poisson also relative to polynomial functions on $\mathfrak{g}_*$.

B. Functions of the form $\varphi \circ \mathbf{J} \in C^\infty(P)$, where $\varphi \in C^\infty(\mathfrak{g}_*)$, are called *collective*. Assuming that all collective functions are in $C^\infty_\omega(P)$, Definition 2.3 implies that the infinitesimally equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}_*$ is a Poisson map relative to the Banach Lie–Poisson structure on $\mathfrak{g}_*$. This follows by simply recalling that the Poisson bracket $\{f, g\}_\omega(p)$ depends only on $\mathbf{d}f(p)$ and $\mathbf{d}g(p)$. Therefore, if $\varphi, \psi \in C^\infty(\mathfrak{g}_*)$, letting $x := \mathbf{d}\varphi(\mathbf{J}(p))$, $y := \mathbf{d}\psi(\mathbf{J}(p)) \in \mathfrak{g}$, we have

$$\{\varphi \circ \mathbf{J}, \psi \circ \mathbf{J}\}_\omega(p) = \{x \circ \mathbf{J}, y \circ \mathbf{J}\}_\omega(p).$$

On the other hand,

$$\begin{aligned}(\{\varphi,\psi\}\circ\mathbf{J})(p) &= \langle[\mathbf{d}\varphi(\mathbf{J}(p)),\mathbf{d}\psi(\mathbf{J}(p))],\mathbf{J}(p)\rangle = \langle[x,y],\mathbf{J}(p)\rangle = \left\langle y, \mathrm{ad}_x^*\mathbf{J}(p)\right\rangle\\ &= -\left\langle y, T_p\mathbf{J}(x_P(p))\right\rangle = -\mathbf{d}(y\circ\mathbf{J})(p)(x_P(p)) = -\mathbf{d}(y\circ\mathbf{J})(p)\,(X_{x\circ\mathbf{J}}(p))\\ &= \{x\circ\mathbf{J}, y\circ\mathbf{J}\}_\omega(p)\end{aligned}$$

by (2.2), which shows that $\{\varphi\circ\mathbf{J},\psi\circ\mathbf{J}\}_\omega = \{\varphi,\psi\}\circ\mathbf{J}$ and hence $\mathbf{J}: P\to\mathfrak{g}_*$ is a Poisson map on elements in $C^\infty_\omega(P)$.

C. Momentum maps often appear through Lie group actions. If $\Phi: G\times P\to P$ is a smooth symplectic action of the Banach Lie group G on the weak symplectic Banach manifold (P,ω), then the infinitesimal generators of the action

$$x_P(p) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Phi(\exp(tx),p),\quad x\in\mathfrak{g}$$

define a smooth Lie algebra action and one can then discuss the existence of the momentum map.

D. Note that $C^\infty_\omega(P)$ is left invariant by the G-action. Indeed, the Hamiltonian vector field of the smooth function $f\circ\Phi_g$ for $f\in C^\infty_\omega(P)$ exists and equals $\Phi_g^* X_f$. Similarly, for any $z\in\mathfrak{g}$, the Hamiltonian vector field of $\mathbf{d}f(z_P)$ exists and equals $[z_P, X_f]$.

E. Propositions 7.3 and 7.4 in [7] show that if the coadjoint isotropy subgroup of $\rho\in\mathfrak{g}_*$ is a closed Lie subgroup of G, the coadjoint orbit is a weak symplectic manifold and the inclusion is a momentum map in the sense of Definition 2.4.

We shall study other momentum maps in subsequent sections.

3. Symplectic induction

The goal of this section is to present the theory of symplectic induction on weak symplectic Banach manifolds. Symplectic induction is a technique that associates to a given Hamiltonian H-space a Hamiltonian G-space whenever H is a Lie subgroup of the Lie group G; see [3,10,11] for various versions of this construction and several applications. We shall formulate this method in the category of Banach manifolds and shall impose also certain splitting assumptions that are satisfied in the examples studied later.

The symplectic induced space. Let G be a Banach Lie group with Banach Lie algebra $\mathfrak{g}$. Let H be a closed Banach Lie subgroup of G with Banach Lie algebra $\mathfrak{h}$. Assume that both $\mathfrak{g}$ and $\mathfrak{h}$ admit preduals $\mathfrak{g}_*$ and $\mathfrak{h}_*$, which are invariant under the coadjoint actions of G and H, respectively (see [7] for various consequences of this assumption). Throughout this section we shall make the following hypotheses:

- $\mathfrak{h}_*\subset\mathfrak{g}_*$,
- there is an Ad^*_H-invariant splitting

$$\mathfrak{g}_* = \mathfrak{h}_*\oplus\mathfrak{h}_*^\perp, \tag{3.1}$$

  where $\mathfrak{h}_*^\perp$ is a Banach Ad^*_H-invariant subspace of $\mathfrak{g}_*$, which means that $\mathrm{Ad}^*_h\mathfrak{h}_*^\perp\subset\mathfrak{h}_*^\perp$ for any $h\in H$, where $\mathrm{Ad}^*: G\to\mathrm{Aut}(\mathfrak{g}_*)$ is the G-coadjoint action,
- $\left(\mathfrak{h}_*^\perp\right)^\circ = \mathfrak{h}$, where $\left(\mathfrak{h}_*^\perp\right)^\circ$ is the annihilator of $\mathfrak{h}_*^\perp$,
- the Banach Lie group H acts symplectically on the weak symplectic Banach manifold (P,ω) and there is a H-equivariant momentum map $\mathbf{J}_P^H: P\to\mathfrak{h}_*$ in the sense of Definition 2.3.

Dualizing the splitting (3.1), we get an Ad_H-invariant splitting

$$\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{h}^\perp, \tag{3.2}$$

where $\mathfrak{h}^\perp := (\mathfrak{h}_*)^\circ$ is the annihilator of the Banach Lie–Poisson space $\mathfrak{h}_*$.

The induction method produces a Hamiltonian G-space by constructing a reduced manifold in the following way. Form the product $P\times G\times\mathfrak{g}_*$ of weak symplectic manifolds, where $G\times\mathfrak{g}_*$ has the weak symplectic form

$$\omega_L(g,\tilde\rho)\left((u_g,\tilde\mu),(v_g,\tilde\nu)\right) = \langle\tilde\nu, T_gL_{g^{-1}}u_g\rangle - \langle\tilde\mu, T_gL_{g^{-1}}v_g\rangle + \langle\tilde\rho,[T_gL_{g^{-1}}u_g, T_gL_{g^{-1}}v_g]\rangle, \tag{3.3}$$

for $g \in G$, $u_g, v_g \in T_gG$, and $\tilde{\rho}, \tilde{\mu}, \tilde{\nu} \in \mathfrak{g}_*$. This formula was introduced in [7] and it looks formally the same as the left trivialized canonical symplectic form on the cotangent bundle of a finite dimensional Lie group (see [1], Section 4.4, Proposition 4.4.1). From (3.3) it follows that

$$C^\infty_{\omega_L}(G \times \mathfrak{g}_*) = \{k \in C^\infty(G \times \mathfrak{g}_*) \mid T_e^* L_g d_1 k(g, \tilde{\rho}) \in \mathfrak{g}_*\},$$

where $d_1 k(g,\rho) \in T_g^* G$ and $d_2 k(g,\rho) \in (\mathfrak{g}_*)^* = \mathfrak{g}$ are the first and second partial derivatives of k. If $k \in C^\infty_{\omega_L}(G \times \mathfrak{g}_*)$, the Hamiltonian vector field $X_k \in \mathfrak{X}(G \times \mathfrak{g}_*)$ has the expression

$$X_k(g, \tilde{\rho}) = \left(T_e L_g d_2 k(g, \tilde{\rho}), \operatorname{ad}^*_{d_2 k(g,\tilde{\rho})} \tilde{\rho} - T_e^* L_g d_1 k(g, \tilde{\rho}) \right). \tag{3.4}$$

Therefore the canonical Poisson bracket of $f, k \in C^\infty_{\omega_L}(G \times \mathfrak{g}_*)$ equals

$$\{f,k\}(g,\rho) = \left\langle d_1 f(g,\rho), T_e L_g d_2 k(g,\rho) \right\rangle - \left\langle d_1 k(g,\rho), T_e L_g d_2 f(g,\rho) \right\rangle - \langle \rho, [d_2 f(g,\rho), d_2 k(g,\rho)] \rangle . \tag{3.5}$$

The left G-action on $G \times \mathfrak{g}_*$ given by $g' \cdot (g,\rho) := (g'g, \rho)$ induces the momentum map $(g,\rho) \mapsto \operatorname{Ad}^*_{g^{-1}} \rho$ which is G-equivariant.

The weak symplectic form $\omega \oplus \omega_L \in \Omega^2(P \times G \times \mathfrak{g}_*)$ is defined by

$$(\omega \oplus \omega_L)(p, g, \tilde{\rho}) \left((a_p, T_e L_g \tilde{x}, \tilde{\mu}), (b_p, T_e L_g \tilde{y}, \tilde{\nu}) \right) = \omega(p)(a_p, b_p) + \langle \tilde{\nu}, \tilde{x} \rangle - \langle \tilde{\mu}, \tilde{y} \rangle + \langle \tilde{\rho}, [\tilde{x}, \tilde{y}] \rangle, \tag{3.6}$$

where $p \in P$, $g \in G$, $\tilde{\rho}, \tilde{\mu}, \tilde{\nu} \in \mathfrak{g}_*$, $\tilde{x}, \tilde{y} \in \mathfrak{g}$, and $a_p, b_p \in T_p P$.

The Banach Lie group H acts on $P \times G \times \mathfrak{g}_*$ by

$$h \cdot (p, g, \tilde{\rho}) := (h \cdot p, gh^{-1}, \operatorname{Ad}^*_{h^{-1}} \tilde{\rho}). \tag{3.7}$$

The infinitesimal generator of this action defined by $z \in \mathfrak{h}$ equals

$$z_{P \times G \times \mathfrak{g}_*}(p, g, \tilde{\rho}) = \left(z_P(p), -T_e L_g z, -\operatorname{ad}^*_z \tilde{\rho} \right)$$

which, by (3.4) and the assumption of the existence of a momentum map induced by the action of H on P, is a Hamiltonian vector field relative to the function $z \circ \left(\mathbf{J}^H_P(p) - \Pi \tilde{\rho} \right)$, where $\Pi : \mathfrak{g}_* \to \mathfrak{h}_*$ is the projection defined by the splitting $\mathfrak{g}_* = \mathfrak{h}_* \oplus \mathfrak{h}_*^\perp$. Therefore, the action (3.7) admits the equivariant momentum map $J^H_{P \times G \times \mathfrak{g}_*} : P \times G \times \mathfrak{g}_* \to \mathfrak{h}_*$ given by

$$\mathbf{J}^H_{P \times G \times \mathfrak{g}_*}(p, g, \tilde{\rho}) = \mathbf{J}^H_P(p) - \Pi \tilde{\rho}. \tag{3.8}$$

The H-action on $P \times G \times \mathfrak{g}_*$ is free and proper because H is a closed Banach Lie subgroup of G. Therefore its restriction to the closed invariant subset $(\mathbf{J}^H_{P \times G \times \mathfrak{g}_*})^{-1}(0)$ is also free and proper. Let us assume now that 0 is a regular value and hence $(\mathbf{J}^H_{P \times G \times \mathfrak{g}_*})^{-1}(0)$ is a submanifold. In concrete applications, such as gravity or Yang-Mills theory, the proof of the regularity of 0 is usually achieved by appealing to elliptic operator theory. With the assumption that 0 is a regular value and that for each $(p, g, \tilde{\rho}) \in (\mathbf{J}^H_{P \times G \times \mathfrak{g}_*})^{-1}(0)$ the map $h \in H \mapsto h \cdot (p, g, \tilde{\rho}) := (h \cdot p, gh^{-1}, \operatorname{Ad}^*_{h^{-1}} \tilde{\rho}) \in (\mathbf{J}^H_{P \times G \times \mathfrak{g}_*})^{-1}(0)$ is an immersion, it follows that the quotient topological space $M := (\mathbf{J}^H_{P \times G \times \mathfrak{g}_*})^{-1}(0)/H$ carries a unique smooth manifold structure relative to which the quotient projection is a submersion. This underlying manifold topology is that of the quotient topological space and it is Hausdorff (see [2], Chapter III, Section 1, Proposition 10 for a proof of these statements). Once these topological conditions are satisfied, a technical lemma (stating that the double symplectic orthogonal of a closed subspace in a weak symplectic Banach space is equal to the original subspace) allows one to extend the original proof of the reduction theorem in finite dimensions (see [5]) to the case of weak symplectic Banach manifolds. We shall not dwell here on these technicalities because in the example of interest to us, treated later, the reduction process will be carried out by hand without any appeal to general theorems. Summarizing, we can form the *induced space* (M, Ω_M) which is a smooth Hausdorff weak symplectic Banach manifold, where Ω_M is the reduced symplectic form on $(\mathbf{J}^H_{P \times G \times \mathfrak{g}_*})^{-1}(0)/H$.

Now note that if we denote $\tilde{\rho} = \rho + \rho^\perp \in \mathfrak{h}_* \oplus \mathfrak{h}_*^\perp$ we get

$$(\mathbf{J}^H_{P \times G \times \mathfrak{g}_*})^{-1}(0) = \left\{ (p, g, \tilde{\rho}) \in P \times G \times \mathfrak{g}_* \mid \mathbf{J}^H_P(p) = \Pi \tilde{\rho} \right\}$$

$$= G \times \left\{ (p, \rho) \in P \times \mathfrak{h}_* \mid \mathbf{J}_P^H(p) = \rho \right\} \times \mathfrak{h}_*^\perp$$
$$\cong G \times P \times \mathfrak{h}_*^\perp,$$

where the H-equivariant diffeomorphism in the last line is given by

$$(p, \rho) \in \left\{ (p, \rho) \in P \times \mathfrak{h}_* \mid \mathbf{J}_P^H(p) = \rho \right\} \longmapsto p \in P.$$

Therefore the weak symplectic Banach manifold $M = (\mathbf{J}^H_{P \times G \times \mathfrak{g}_*})^{-1}(0)/H$ is diffeomorphic to the fiber bundle $G \times_H (P \times \mathfrak{h}_*^\perp) \to G/H$ associated to $G \to G/H$.

The weak symplectic form on the induced space. Denote by $\pi_0 : G \times P \times \mathfrak{h}_*^\perp \to G \times_H (P \times \mathfrak{h}_*^\perp) =: M$ the projection onto the H-orbit space. The next statement gives the weak symplectic form on M.

Proposition 3.1. *The total space of the associated fiber bundle $G \times_H (P \times \mathfrak{h}_*^\perp) \to G/H$ has a weak symplectic form Ω given by*

$$\begin{aligned}
&\Omega(\pi_0(g, p, \rho^\perp)) \left(T_{(g,p,\rho^\perp)}\pi_0(T_e L_g(x + x^\perp), a_p, \mu^\perp), T_{(g,p,\rho^\perp)}\pi_0(T_e L_g(y + y^\perp), b_p, \nu^\perp) \right) \\
&\quad = \omega(p)(a_p, b_p) + \left\langle T_p\mathbf{J}_P^H(b_p), x \right\rangle + \left\langle \nu^\perp, x^\perp \right\rangle - \left\langle T_p\mathbf{J}_P^H(a_p), y \right\rangle - \left\langle \mu^\perp, y^\perp \right\rangle \\
&\qquad + \left\langle \mathbf{J}_P^H(p), [x, y] \right\rangle + \left\langle \rho^\perp, [x^\perp, y] + [x, y^\perp] \right\rangle + \left\langle \mathbf{J}_P^H(p) + \rho^\perp, [x^\perp, y^\perp] \right\rangle,
\end{aligned} \tag{3.9}$$

for $g \in G$, $p \in P$, $\rho^\perp, \mu^\perp, \nu^\perp \in \mathfrak{h}_*^\perp$, $x, y \in \mathfrak{h}$, $x^\perp, y^\perp \in \mathfrak{h}^\perp$, and $a_p, b_p \in T_pP$. Equivalently, using on the right hand side only tangent vectors of the form

$$\left(a_p - x_P(p), T_e L_g(2x + x^\perp), \mu^\perp + \mathrm{ad}_x^* \rho^\perp \right)$$

which are transversal to the H-orbits in the zero level set of the momentum map and hence represent the tangent space $T_{\pi_0(g,p,\rho^\perp)}M$ to the reduced manifold M, the expression of Ω is

$$\begin{aligned}
&\Omega(\pi_0(g, p, \rho^\perp)) \left(T_{(g,p,\rho^\perp)}\pi_0(T_e L_g(x + x^\perp), a_p, \mu^\perp), T_{(g,p,\rho^\perp)}\pi_0(T_e L_g(y + y^\perp), b_p, \nu^\perp) \right) \\
&\quad = \omega(p)(a_p - x_P(p), b_p - y_P(p)) + \left\langle T_p\mathbf{J}_P^H(b_p - y_P(p)), 2x \right\rangle + \left\langle \nu^\perp + \mathrm{ad}_y^* \rho^\perp, x^\perp \right\rangle \\
&\qquad - \left\langle T_p\mathbf{J}_P^H(a_p - x_P(p)), 2y \right\rangle - \left\langle \mu^\perp + \mathrm{ad}_x^* \rho^\perp, y^\perp \right\rangle + \left\langle \mathbf{J}_P^H(p), [2x, 2y] \right\rangle \\
&\qquad + \left\langle \rho^\perp, [x^\perp, 2y] + [2x, y^\perp] \right\rangle + \left\langle \mathbf{J}_P^H(p) + \rho^\perp, [x^\perp, y^\perp] \right\rangle.
\end{aligned} \tag{3.10}$$

Proof. We begin with the proof (3.9). Let $i_0 : G \times P \times \mathfrak{h}_*^\perp \hookrightarrow P \times G \times \mathfrak{g}_*$ be the inclusion $i_0(g, p, \rho^\perp) := (p, g, \mathbf{J}_P^H(p) + \rho^\perp)$. For $p \in P$, $\rho^\perp, \mu^\perp, \nu^\perp \in \mathfrak{h}_*^\perp$, $g \in G$, $\tilde{x} = x + x^\perp$, $\tilde{y} = y + y^\perp \in \mathfrak{g}$, $x, y \in \mathfrak{h}$, $x^\perp, y^\perp \in \mathfrak{h}^\perp$, and $a_p, b_p \in T_pP$, the reduction theorem and (3.6) give

$$\begin{aligned}
&\Omega(\pi_0(g, p, \rho^\perp)) \left(T_{(g,p,\rho^\perp)}\pi_0(T_e L_g \tilde{x}, a_p, \mu^\perp), T_{(g,p,\rho^\perp)}\pi_0(T_e L_g \tilde{y}, b_p, \nu^\perp) \right) \\
&\quad = i_0^*(\omega \oplus \omega_L)(p, g, \rho^\perp) \left((a_p, T_e L_g \tilde{x}, \mu^\perp), (b_p, T_e L_g \tilde{y}, \nu^\perp) \right) \\
&\quad = (\omega \oplus \omega_L)(p, g, \mathbf{J}_P^H(p) + \rho^\perp) \left((a_p, T_e L_g \tilde{x}, T_p\mathbf{J}_P^H(a_p) + \mu^\perp), (b_p, T_e L_g \tilde{y}, T_p\mathbf{J}_P^H(b_p) + \nu^\perp) \right) \\
&\quad = \omega(p)(a_p, b_p) + \left\langle T_p\mathbf{J}_P^H(b_p) + \nu^\perp, x + x^\perp \right\rangle - \left\langle T_p\mathbf{J}_P^H(a_p) + \mu^\perp, y + y^\perp \right\rangle \\
&\qquad + \left\langle \mathbf{J}_P^H(p) + \rho^\perp, [x + x^\perp, y + y^\perp] \right\rangle.
\end{aligned}$$

Since $[x + x^\perp, y + y^\perp] = [x, y] + [x^\perp, y] + [x, y^\perp] + [x^\perp, y^\perp]$, $[x, y] \in \mathfrak{h} = (\mathfrak{h}_*^\perp)^\circ$, $[x^\perp, y] + [x, y^\perp] \in \mathfrak{h}^\perp = (\mathfrak{h}_*)^\circ$ (because the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ is Ad_H^*-invariant), $\rho^\perp \in \mathfrak{h}_*^\perp$, and $\mathbf{J}_P^H(p) \in \mathfrak{h}_*$, the last term becomes

$$\left\langle \mathbf{J}_P^H(p)+\rho^\perp, [x+x^\perp, y+y^\perp]\right\rangle = \left\langle \mathbf{J}_P^H(p), [x,y]\right\rangle + \left\langle \rho^\perp, [x^\perp, y]+[x, y^\perp]\right\rangle + \left\langle \mathbf{J}_P^H(p)+\rho^\perp, [x^\perp, y^\perp]\right\rangle.$$

Since $T_p\mathbf{J}_P^H(b_p) \in \mathfrak{h}_*$, $\nu^\perp \in \mathfrak{h}_*^\perp$, $x \in \mathfrak{h} = (\mathfrak{h}_*^\perp)^\circ$, and $x^\perp \in \mathfrak{h}^\perp = (\mathfrak{h}_*)^\circ$, the second term becomes

$$\left\langle T_p\mathbf{J}_P^H(b_p)+\nu^\perp, x+x^\perp\right\rangle = \left\langle T_p\mathbf{J}_P^H(b_p), x\right\rangle + \left\langle \nu^\perp, x^\perp\right\rangle.$$

Similarly, the third term is

$$\left\langle T_p\mathbf{J}_P^H(a_p)+\mu^\perp, y+y^\perp\right\rangle = \left\langle T_p\mathbf{J}_P^H(a_p), y\right\rangle + \left\langle \mu^\perp, y^\perp\right\rangle.$$

Thus we get

$$\begin{aligned} &\Omega(\pi_0(g,p,\rho^\perp))\left(T_{(g,p,\rho^\perp)}\pi_0(T_eL_g(x+x^\perp), a_p, \mu^\perp), T_{(g,p,\rho^\perp)}\pi_0(T_eL_g(y+y^\perp), b_p, \nu^\perp)\right) \\ &\quad = \omega(p)(a_p, b_p) + \left\langle T_p\mathbf{J}_P^H(b_p), x\right\rangle + \left\langle \nu^\perp, x^\perp\right\rangle - \left\langle T_p\mathbf{J}_P^H(a_p), y\right\rangle - \left\langle \mu^\perp, y^\perp\right\rangle \\ &\qquad + \left\langle \mathbf{J}_P^H(p), [x,y]\right\rangle + \left\langle \rho^\perp, [x^\perp, y]+[x, y^\perp]\right\rangle + \left\langle \mathbf{J}_P^H(p)+\rho^\perp, [x^\perp, y^\perp]\right\rangle \end{aligned}$$

which proves (3.9).

We want to simplify this expression by taking advantage of the H-action on the zero level set of the momentum map. For $x \in \mathfrak{h}$ we have by H-equivariance of $\mathbf{J}_P^H$ and the Ad_H^*-invariance of the splitting $\mathfrak{g}_* = \mathfrak{h}_* \oplus \mathfrak{h}_*^\perp$

$$\begin{aligned} x_{P\times G\times \mathfrak{g}_*}(p, g, \mathbf{J}_P^H(p)+\rho^\perp) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\exp tx\cdot p, g\exp(-tx), \mathrm{Ad}^*_{\exp(-tx)}(\mathbf{J}_P^H(p)+\rho^\perp)\right) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\exp tx\cdot p, g\exp(-tx), \mathbf{J}_P^H(\exp tx\cdot p)+\mathrm{Ad}^*_{\exp(-tx)}\rho^\perp\right) \\ &= \left(x_P(p), -T_eL_gx, T_p\mathbf{J}_P^H(x_P(p)) - \mathrm{ad}_x^*\rho^\perp\right). \end{aligned}$$

Now decompose

$$\begin{aligned} \left(a_p, T_eL_g(x+x^\perp), T_p\mathbf{J}_P^H(a_p)+\mu^\perp\right) &= \left(x_P(p), -T_eL_gx, T_p\mathbf{J}_P^H(x_P(p)) - \mathrm{ad}_x^*\rho^\perp\right) \\ &\quad + \left(a_p - x_P(p), T_eL_g(2x+x^\perp), T_p\mathbf{J}_P^H(a_p - x_P(p)) + \mu^\perp + \mathrm{ad}_x^*\rho^\perp\right). \end{aligned}$$

Since the form Ω does not depend on the first summand, this means that we can replace everywhere in (3.9) a_p by $a_p - x_P(p)$, x by $2x$, and $\mu^\perp$ by $\mu^\perp + \mathrm{ad}_x^*\rho^\perp$. Similarly, we can replace b_p by $b_p - y_P(p)$, y by $2y$, and $\nu^\perp$ by $\nu^\perp + \mathrm{ad}_y^*\rho^\perp$. Thus (3.9) becomes

$$\begin{aligned} &\omega(p)(a_p - x_P(p), b_p - y_P(p)) + \left\langle T_p\mathbf{J}_P^H(b_p - y_P(p)), 2x\right\rangle + \left\langle \nu^\perp + \mathrm{ad}_y^*\rho^\perp, x^\perp\right\rangle \\ &\quad - \left\langle T_p\mathbf{J}_P^H(a_p - x_P(p)), 2y\right\rangle - \left\langle \mu^\perp + \mathrm{ad}_x^*\rho^\perp, y^\perp\right\rangle + \left\langle \mathbf{J}_P^H(p), [2x, 2y]\right\rangle \\ &\quad + \left\langle \rho^\perp, [x^\perp, 2y]+[2x, y^\perp]\right\rangle + \left\langle \mathbf{J}_P^H(p)+\rho^\perp, [x^\perp, y^\perp]\right\rangle \end{aligned}$$

which proves (3.10). ■

Remark. If $H = G$, then one can verify directly that the map $\Psi : G\times_H(P\times\{0\}) \to P$ given by $\Psi(\pi_0(g,p,0)) := g\cdot p$ is a diffeomorphism between the weak symplectic manifolds $(G\times_H(P\times\{0\}), \Omega)$ (the induced space) and (P, ω) (the original manifold).

The momentum map on the induced space. Now we shall construct a G-action on the induced space $\left(G\times_H(P\times\mathfrak{h}_*^\perp), \Omega\right)$ and a G-equivariant momentum map $\mathbf{J}_M^G : G\times_H(P\times\mathfrak{h}_*^\perp) \to \mathfrak{g}_*$.

The Banach Lie group G acts on $G \times P \times \mathfrak{h}_*^\perp$ by $g' \cdot (g, p, \rho^\perp) := (g'g, p, \rho^\perp)$. This G-action commutes with the H-action and so G acts on the induced space $G \times_H (P \times \mathfrak{h}_*^\perp)$ by $g' \cdot [g, p, \rho^\perp] := [g'g, p, \rho^\perp]$. It is routine to verify that this action preserves the weak symplectic form Ω and that the map

$$\mathbf{J}_M^G([g, p, \rho^\perp]) = \mathrm{Ad}_{g^{-1}}^* \left(\mathbf{J}_P^H(p) + \rho^\perp\right) \tag{3.11}$$

satisfies the conditions of Definition 2.3. We conclude hence the following result.

Proposition 3.2. *The map $\mathbf{J}_M^G : G \times_H (P \times \mathfrak{h}_*^\perp) \to \mathfrak{g}_*$ given by (3.11) is a G-equivariant momentum map for the G-weak symplectic manifold $\left(G \times_H (P \times \mathfrak{h}_*^\perp), \Omega\right)$.*

The goal of the induction construction has now been achieved: starting with the Hamiltonian H-space (P, ω), where H is a closed Lie subgroup of a Lie group G, a new Hamiltonian G-space has been constructed, namely $(G \times_H (P \times \mathfrak{h}_*^\perp), \Omega)$.

4. Banach Lie–Poisson spaces of k-diagonal trace class operators

In this section we introduce families of trace class operators that will be useful later on. For all the proofs of the statements below we refer to [8].

Let us begin with some useful preliminary notation. By $\{|n\rangle\}_{n=0}^\infty$ we denote an orthonormal basis of the real separable Hilbert space $\mathcal{H}$. It induces the Schauder basis $\{|n\rangle\langle m|\}_{n,m=0}^\infty$ of L^1. Thus, for $\rho \in L^1$ one has the expression

$$\rho = \sum_{n,m=0}^{\infty} \rho_{nm} |n\rangle\langle m|, \tag{4.1}$$

where the series is convergent in the $\|\cdot\|_1$-topology. Similarly, for $x \in L^\infty$, one has

$$x = \sum_{l,k=0}^{\infty} x_{lk} |l\rangle\langle k|, \tag{4.2}$$

where the series is convergent in the w^*-topology. In particular, the shift operator $S \in L^\infty$ and its adjoint S^T have the expressions

$$S := \sum_{n=0}^{\infty} |n\rangle\langle n+1| \quad \text{and its adjoint } S^T := \sum_{n=0}^{\infty} |n+1\rangle\langle n|. \tag{4.3}$$

Let $L_0^\infty := \left\{\sum_{n=0}^\infty x_n |n\rangle\langle n| \mid \{x_n\}_{n=0}^\infty \in \ell^\infty\right\}$ and $L_0^1 := \left\{\sum_{n=0}^\infty \rho_n |n\rangle\langle n| \mid \{\rho_n\}_{n=0}^\infty \in \ell^1\right\}$ denote the Banach subspaces of diagonal bounded and trace class operators, respectively. In subsequent considerations we will be interested in the following Banach subspaces:

- $L_+^\infty := \left\{x = \sum_{n=0}^\infty x_n S^n \mid x_n \in L_0^\infty\right\}$
- $L_-^1 := \left\{\rho = \sum_{n=0}^\infty (S^T)^n \rho_n \mid \rho_n \in L_0^1\right\}$
- $L_{+,k}^\infty := \left\{x = \sum_{n=0}^{k-1} x_n S^n \mid x_n \in L_0^\infty\right\}$, for $k \geq 2$
- $L_{-,k}^1 := \left\{\rho = \sum_{n=0}^{k-1} (S^T)^n \rho_n \mid \rho_n \in L_0^1\right\}$, for $k \geq 2$
- $I_{+,k}^\infty := \left\{x = \sum_{n=k}^\infty x_n S^n \mid x_n \in L_0^\infty\right\}$, for $k \geq 1$
- $I_{-,k}^1 := \left\{\rho = \sum_{n=k}^\infty (S^T)^n \rho_n \mid \rho_n \in L_0^1\right\}$, for $k \geq 1$
- $B_{+,k}^\infty := \left\{x = x_0 + x_{k-1} S^{k-1} \mid x_0, x_{k-1} \in L_0^\infty\right\}$, for $k \geq 2$
- $B_{-,k}^1 := \left\{\rho = \rho_0 + (S^{k-1})^T \rho_{k-1} \mid \rho_0, \rho_{k-1} \in L_0^1\right\}$, for $k \geq 2$
- $(B_{+,k}^\infty)^\perp := \left\{\rho = x_1 S + \cdots + x_{k-2} S^{k-2} \mid x_i \in L_0^\infty\right\}$, for $k \geq 3$
- $(B_{-,k}^1)^\perp := \left\{\rho = S^T \rho_1 + \cdots + (S^T)^{k-2} \rho_{k-2} \mid \rho_i \in L_0^1\right\}$, for $k \geq 3$
- $L_k^\infty := L_0^\infty S^k$, for $k \geq 0$

- $L^1_k := (S^T)^k L^1_0$, for $k \geq 0$.

In this list the every space of trace class operators is predual to the space of bounded operators above it, the isomorphism being given by the duality pairing (2.12). So, for example, the predual of $B^\infty_{+,k}$ is $B^1_{-,k}$. Note that the subalgebra $B^\infty_{+,k}$ of $L^\infty_{+,k}$ is formed by upper triangular bounded operators that have only two non-zero diagonals, namely the main diagonal and the strictly upper $k-1$ diagonal. A symmetric remark applies to $B^1_{-,k}$.

The space of upper triangular bounded operators L^∞_+ is a Banach Lie algebra whose underlying Banach Lie group is $GL^\infty_+ := GL^\infty \cap L^\infty_+$, where GL^∞ is the Banach Lie group of invertible bounded operators. The Banach Lie group

$$\begin{aligned} GI^\infty_{+,k} &:= (\mathbb{I} + I^\infty_{+,k}) \cap GL^\infty_+ \\ &= \{\mathbb{I} + \varphi \mid \varphi \in I^\infty_{+,k}, \mathbb{I} + \varphi \text{ is invertible in } GL^\infty_+\} \end{aligned} \tag{4.4}$$

has $I^\infty_{+,k}$ as its Banach Lie algebra. Since $I^\infty_{+,k}$ is an ideal in L^∞_+ (relative to both the associative and Lie structures), $GI^\infty_{+,k}$ is a closed normal Banach Lie subgroup in GL^∞_+ and the factor group $GL^\infty_+/GL^\infty_{+,k}$ is a Banach Lie group isomorphic to the group

$$GL^\infty_{+,k} = \left\{ g = \sum_{i=0}^{k-1} g_i S^i \mid g_i \in L^\infty_0, |g_0| \geq \varepsilon(g_0)\mathbb{I} \text{ for some } \varepsilon(g_0) > 0 \right\} \tag{4.5}$$

whose multiplication is defined by

$$g \circ_k h := \sum_{l=0}^{k-1} \left(\sum_{i=0}^{l} g_i s^i (h_{l-i}) \right) S^l, \tag{4.6}$$

and the inverse $g^{-1} = g_0^{-1} + h_1 S + \cdots + h_{k-1} S^{k-1}$ of $g = g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} \in GL^\infty_{+,k}$ is given by

$$h_p = -g_0^{-1} \left[\sum_{r=1}^{p-1} \sum (-1)^{r-1} g_{i_1} s^{j_1}(g_0^{-1} g_{i_2}) \dots s^{j_q}(g_0^{-1} g_{i_q}) \dots s^{j_r}(g_0^{-1} g_{i_r}) \right] s^p(g_0^{-1}), \tag{4.7}$$

$1 \leq p \leq k-1$, where the second sum is taken over all indices $\{i_1, \dots, i_r, j_1, \dots, j_r\}$ such that $i_1 + \cdots + i_r = p$ (equality between the i_q is permitted), $0 \leq i_1, \dots, i_r \leq p$, $1 \leq i_1 = j_1 < j_2 < \cdots < j_r = p - i_r \leq p-1$. In these formulas $s, \tilde{s} : L^\infty_0 \to L^\infty_0$ ($s, \tilde{s} : L^1_0 \to L^1_0$) are given by

$$\left. \begin{array}{lll} Sx = s(x)S & \text{or} & xS^T = S^T s(x) \\ S^T x = \tilde{s}(x) S^T & \text{or} & xS = S\tilde{s}(x) \end{array} \right\} \tag{4.8}$$

for $x \in L^\infty_0$ or $x \in L^1_0$. Thus, $s(x_0, x_1, x_2 \dots, x_n, \dots) := (x_1, x_2, \dots, x_n, \dots)$ and $\tilde{s}(x_0, x_1, x_2 \dots, x_n, \dots) := (0, x_0, x_1, x_2, \dots, x_n, \dots)$ for any $(x_0, x_1, x_2 \dots, x_n, \dots) \in \ell^\infty \cong L^\infty_0$; s and $\tilde{s}$ are mutually adjoint operators.

The Banach Lie algebra of $GL^\infty_{+,k}$ is $L^\infty_{+,k}$ with the bracket defined by

$$[x, y]_k := x \circ_k y - y \circ_k x = \sum_{l=0}^{k-1} \sum_{i=0}^{l} \left(x_i s^i (y_{l-i}) - y_i s^i (x_{l-i}) \right) S^l \tag{4.9}$$

Since $\left(L^\infty_{+,k}\right)_* = L^1_{-,k}$, the Lie–Poisson bracket on $L^1_{-,k}$ assumes the following form

$$\begin{aligned} \{f, g\}_k(\rho) &= \operatorname{Tr}\left(\rho \left[Df(\rho), Dg(\rho)\right]_k\right) \\ &= \sum_{l=0}^{k-1} \sum_{i=0}^{l} \operatorname{Tr}\left[\rho_l \left(\frac{\delta f}{\delta \rho_i}(\rho) s^i \left(\frac{\delta g}{\delta \rho_{l-i}}(\rho) \right) - \frac{\delta g}{\delta \rho_i}(\rho) s^i \left(\frac{\delta f}{\delta \rho_{l-i}}(\rho) \right) \right) \right] \end{aligned} \tag{4.10}$$

for $f, g \in C^\infty(L^1_{-,k})$, where $\frac{\delta f}{\delta \rho_i}(\rho)$ denotes the partial functional derivative of f relative to ρ_i defined by $Df(\rho) = \frac{\delta f}{\delta \rho_0}(\rho) + \frac{\delta f}{\delta \rho_1}(\rho) S + \cdots + \frac{\delta f}{\delta \rho_{k-1}}(\rho) S^{k-1}$. If $k = \infty$ we get the Lie–Poisson bracket on L^1_-.

The coadjoint action $\left(\mathrm{Ad}^{+,k}\right)^*$ of $GL^{\infty}_{+,k}$ on $L^1_{-,k}$ is given by

$$(\mathrm{Ad}^{+,k})^*_{g^{-1}}\rho = \sum_{i,j,l=0,\ j\geq i+l}^{k-1} (S^T)^{j-i-l}\tilde{s}^l[s^j(\tilde{s}^i(g_i))\rho_j h_l], \tag{4.11}$$

where $\rho = \rho_0 + S^T\rho_1 + \cdots + (S^T)^{k-1}\rho_{k-1} \in L^1_{-,k}$, $g = g_0 + g_1 S + \cdots + g_{k-1}S^{k-1} \in GL^{\infty}_{+,k}$, and the diagonal operators h_l are expressed in terms of the g_i in (4.7).

Thus, the Hamiltonian equations defined by $h \in C^{\infty}(L^1_{-,k})$ are

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_j = -\sum_{l=j}^{k-1}\left(\tilde{s}^{l-j}\left(\rho_l \frac{\delta h_k}{\delta\rho_{l-j}}\right) - \rho_l s^j\left(\frac{\delta h_k}{\delta\rho_{l-j}}\right)\right) \quad \text{for } j = 0, 1, 2, \ldots, k-1. \tag{4.12}$$

The functions $I^k_l \in C^{\infty}(L^1_{-,k})$ defined by

$$I^k_l(\rho) := \frac{1}{l}\operatorname{Tr}\left(\rho + \rho^T - \rho_0\right)^l, \quad \text{for } l \in \mathbb{N}, \tag{4.13}$$

where $\rho = \sum_{n=0}^{k-1}(S^T)^n\rho_n \in L^1_{-,k}$, are in involution with respect to the Poisson bracket (4.10), that is, $\{I^k_n, I^k_m\}_k = 0$ for all $n, m \in \mathbb{N}$. For $k = 2$ they give the Toda hierarchy. So one can consider the Hamiltonian system described by (4.12) and (4.13) as a k-diagonal version of the semi-infinite Toda lattice.

Now let us fix the Banach Lie subgroup $GB^{\infty}_{+,k} \subset GL^{\infty}_{+,k}$ of bidiagonal elements $g = g_0 + gS^{k-1} \in GL^{\infty}_{+,k}$. The group multiplication (4.6) takes on a simple form on $GB^{\infty}_{+,k}$, namely,

$$g \circ_k h = g_0h_0 + (g_0h_{k-1} + g_{k-1}s^{k-1}(h_0))S^{k-1} \tag{4.14}$$

and the inverse of g in $GB^{\infty}_{+,k}$ is given by

$$g^{-1} = g_0^{-1} - g_0^{-1}g_{k-1}s^{k-1}(g_0^{-1})S^{k-1}. \tag{4.15}$$

The Lie bracket of $x, y \in B^{\infty}_{+,k}$ has the expression

$$[x, y]_k = \left(x_{k-1}(s^{k-1}(y_0) - y_0) - y_{k-1}(s^{k-1}(x_0) - x_0)\right)S^{k-1}. \tag{4.16}$$

The group coadjoint action $\left(\mathrm{Ad}^{+,k}\right)^*_{g^{-1}} : B^1_{-,k} \to B^1_{-,k}$ for $g := g_0 + g_{k-1}S^{k-1} \in GB^{\infty}_{+,k} \subset GL^{\infty}_{+,k}$ and Lie algebra coadjoint action $(\mathrm{ad}^{+,k})^*_x : B^1_{-,k} \to B^1_{-,k}$, for $x := x_0 + x_{k-1}S^{k-1} \in B^{\infty}_{+,k} \subset L^{\infty}_{+,k}$ are given by

$$\begin{aligned}\left(\mathrm{Ad}^{+,k}\right)^*_{g^{-1}}\rho &= \rho_0 + g_0^{-1}g_{k-1}\rho_{k-1} - \tilde{s}^{k-1}\left(g_0^{-1}g_{k-1}\rho_{k-1}\right)\left(\mathbb{I} - \sum_{j=0}^{k-2}p_j\right)\\ &\quad + \left(S^T\right)^{k-1}s^{k-1}(g_0)g_0^{-1}\rho_{k-1}\end{aligned} \tag{4.17}$$

and

$$\left(\mathrm{ad}^{+,k}\right)^*_x\rho = \tilde{s}^{k-1}(\rho_{k-1}x_{k-1}) - \rho_{k-1}x_{k-1} + \left(S^T\right)^{k-1}\rho_{k-1}(x_0 - s^{k-1}(x_0)) \tag{4.18}$$

where $\rho := \rho_0 + (S^T)^{k-1}\rho_{k-1} \in I^1_{-,0,k-1}$.

The Lie algebra $B^{\infty}_{+,k}$ of $GB^{\infty}_{+,k}$ has $B^1_{-,k}$ as predual. One has the Banach space splittings

$$L^{\infty}_{+,k} = B^{\infty}_{+,k} \oplus (B^{\infty}_{+,k})^{\perp} \tag{4.19}$$

and

$$L^1_{-,k} = B^1_{-,k} \oplus (B^1_{-,k})^{\perp}. \tag{4.20}$$

We shall prove that the splittings (4.19) and (4.20) are $\mathrm{Ad}_{GB^{\infty}_{+,k}}$ and $\mathrm{Ad}^*_{GB^{\infty}_{+,k}}$-invariant, respectively.

Let us show that the splitting (4.19) is invariant relative to the restriction $\mathrm{Ad}_{GB^{\infty}_{+,k}}$ of the adjoint action $\mathrm{Ad}^{+,k}$ of the Banach Lie group $GL^{\infty}_{+,k}$ to the Lie subgroup $GB^{\infty}_{+,k}$. Clearly the summand $B^{\infty}_{+,k}$ is preserved because it is the Lie algebra of $GB^{\infty}_{+,k}$. To see that the second factor $(B^{\infty}_{+,k})^{\perp}$ is also preserved, using (4.15), it suffices to show that for any $h = h_0 + h_{k-1}S^{k-1} \in GB^{\infty}_{+,k}$ and any $x_1 S + \cdots + x_{k-2}S^{k-2} \in (B^{\infty}_{+,k})^{\perp}$ we have

$$\begin{aligned}
&(\mathrm{Ad}^{+,k})_h(x_1 S + \cdots x_{k-2}S^{k-2})\\
&= \left(h_0 + h_{k-1}S^{k-1}\right) \circ_k \left(x_1 S + \cdots + x_{k-2}S^{k-2}\right) \circ_k \left(h_0^{-1} - h_0^{-1}h_{k-1}s^{k-1}(h_0^{-1})S^{k-1}\right)\\
&= h_0 s(h_0^{-1})x_1 S + \cdots + h_0 s^{k-2}(h_0^{-1})x_{k-2}S^{k-2}
\end{aligned} \tag{4.21}$$

which is a straightforward verification.

Next we show that the splitting (4.20) is invariant relative to the restriction $\mathrm{Ad}^*_{GB^{\infty}_{+,k}}$ of the coadjoint action $(\mathrm{Ad}^{+,k})^*$ of $GL^{\infty}_{+,k}$ to the Lie subgroup $GB^{\infty}_{+,k}$. First, by (4.17) the $GB^{\infty}_{+,k}$ coadjoint action preserves the predual $B^1_{-,k}$. Second, to show that the second summand $(B^1_{-,k})^{\perp}$ is also preserved, one verifies directly, using (4.15), that for any $h = h_0 + h_{k-1}S^{k-1} \in GB^{\infty}_{+,k}$ and $S^T\rho_1 + \cdots + (S^T)^{k-2}\rho_{k-2} \in (B^1_{-,k})^{\perp}$ we have

$$(\mathrm{Ad}^{+,k})^*_{h^{-1}}(S^T\rho_1 + \cdots + (S^T)^{k-2}\rho_{k-2}) = S^T s(h_0)h_0^{-1}\rho_1 + \cdots + (S^T)^{k-2}s^{k-2}(h_0)h_0^{-1}\rho_{k-2}. \tag{4.22}$$

In order to satisfy all hypotheses necessary for the symplectic induction procedure (see Section 3) we define:

(i) the map $\mathcal{J}_\nu : (\ell^{\infty} \times \ell^1, \omega) \to (B^{\infty}_{+,k}, \{\,,\})$ by

$$\mathcal{J}_\nu(\mathbf{q}, \mathbf{p}) = \mathbf{p} + (S^T)^{k-1}\nu e^{s^{k-1}(\mathbf{q})-\mathbf{q}}, \tag{4.23}$$

where the fixed element $(S^T)^{k-1}\nu \in L^1_{-,k}$ satisfies $\nu_{ii} \neq 0$ for all $i = 0, 1, 2, \ldots$;

(ii) the ν-dependent symplectic action of $GB^{\infty}_{+,k}$ on $\ell^{\infty} \times \ell^1$ by

$$\boldsymbol{\sigma}^{\nu}_g(\mathbf{q}, \mathbf{p}) := \left(\mathbf{q} + \log g_0, \mathbf{p} + g_{k-1}g_0^{-1}\nu e^{s^{k-1}(\mathbf{q})-\mathbf{q}} - \tilde{s}^{k-1}\left(g_{k-1}g_0^{-1}\nu e^{s^{k-1}(\mathbf{q})-\mathbf{q}}\right)\right), \tag{4.24}$$

where $g := g_0 + g_{k-1}S^{k-1} \in GB^{\infty}_{+,k}$ and $(\mathbf{q}, \mathbf{p}) \in \ell^{\infty} \times \ell^1$.

Here and in what follows the logarithm and the exponential of a sequence is the sequence whose elements are the logarithms and the exponentials of every element in the sequence. Denote by $GL^{\infty,k-1}_0$ the Banach Lie subgroup of $(k-1)$-periodic elements of GL^{∞}_0, that is, $g_0 \in GL^{\infty,k-1}_0$ if and only if $s^{k-1}(g_0) = g_0$. Denote by $L^{\infty,k-1}_0$ the Banach Lie algebra of $GL^{\infty,k-1}_0$. In [8] we proved the following.

Proposition 4.1. *The smooth map $\mathcal{J}_\nu : \ell^{\infty} \times \ell^1 \to B^1_{-,k}$ given by (4.23) is constant on the $\boldsymbol{\sigma}^{\nu}$-orbits of the subgroup $GL^{\infty,k-1}_0$. In addition:*

(i) *$\mathcal{J}_\nu$ is a momentum map. More precisely, $\{f \circ \mathcal{J}_\nu, g \circ \mathcal{J}_\nu\}_\omega = \{f, g\}_{0,k-1} \circ \mathcal{J}_\nu$, for all $f, g \in C^{\infty}(B^1_{-,k})$, where $\{\cdot,\cdot\}_\omega$ is the canonical Poisson bracket of the weak symplectic Banach space $\left(\ell^{\infty} \times \ell^1, \omega\right)$ given by (2.10) and $\{\,,\}_{0,k-1}$ is the Lie–Poisson bracket on $B^1_{-,k}$ given by*

$$\{f, h\}_{B^1_{-,k}}(\rho) = \mathrm{Tr}\left[\rho_{k-1}\left(\frac{\partial f}{\partial \rho_{k-1}}\left(s^{k-1}\left(\frac{\partial h}{\partial \rho_0}\right) - \frac{\partial h}{\partial \rho_0}\right) - \frac{\partial h}{\partial \rho_{k-1}}\left(s^{k-1}\left(\frac{\partial f}{\partial \rho_0}\right) - \frac{\partial f}{\partial \rho_0}\right)\right)\right] \tag{4.25}$$

(ii) *$\mathcal{J}_\nu$ is $GB^{\infty}_{+,k}$-equivariant, that is, $\mathcal{J}_\nu \circ \boldsymbol{\sigma}^{\nu}_g = \left(\mathrm{Ad}^{-,k}\right)^*_{g^{-1}} \circ \mathcal{J}_\nu$ for any $g \in GB^{\infty}_{+,k}$.*

Now we are ready to implement a symplectic induction procedure.

5. The momentum map for $((\ell^\infty)^{k-1} \times (\ell^1)^{k-1}, \Omega_k)$

In this section we construct a weak symplectic form Ω_k on $(\ell^\infty)^{k-1} \times \left(\ell^1\right)^{k-1}$ (see (5.10)) which has a non-canonical term responsible for the interaction of the Toda system with some kind of an external "field". Then we find a $GL^\infty_{+,k}$-equivariant momentum map $\mathbf{J}_k : (\ell^\infty)^{k-1} \times \left(\ell^1\right)^{k-1} \to L^1_{-,k}$ (see (5.5)) which can be interpreted as a generalization of the momentum map (4.23) defined for the bidiagonal case. We shall illustrate the hierarchy of dynamical systems obtained in this way by presenting the special case $k = 3$ in detail (see (5.19)). The simpler case $k = 2$ does not add anything new since one recovers by the symplectic induction method the original semi-infinite Toda system studied in [8].

We shall apply the induction method discussed in Section 3 to the weak symplectic manifold $(P, \omega) = (\ell^\infty \times \ell^1, \omega)$ with ω given by (2.6), the Banach Lie group $G := (GL^\infty_{+,k}, \circ_k)$ defined in (4.5), and the Banach Lie subgroup $H := GB^\infty_{+,k}$. As will be seen, the abstract constructions presented in Section 3 become completely explicit in this case.

We begin by listing the objects involved in this construction. The Banach Lie algebra is $\mathfrak{g} := L^\infty_{+,k}$, the subalgebra is $\mathfrak{h} := B^\infty_{+,k}$, and its closed split complement is $\mathfrak{h}^\perp := (B^\infty_{+,k})^\perp$. At the level of the preduals we have $\mathfrak{g}_* = L^1_{-,k}$, $\mathfrak{h}_* = B^1_{-,k}$, and its closed split complement $\mathfrak{h}^\perp_* = (B^1_{-,k})^\perp$. We have hence the adjoint and coadjoint invariant Banach space direct sums (4.19) and (4.20). Thus, every $\rho \in L^1_{-,k}$ decomposes uniquely as $\rho = \gamma + \gamma^\perp$, where $\gamma \in B^1_{-,k}$ and $\gamma^\perp \in (B^1_{-,k})^\perp$.

We fix in all considerations below an element $\nu \in L^1_0$. According to the general theory we shall take the weak symplectic manifolds $GL^\infty_{+,k} \times L^1_{-,k}$ and $\ell^\infty \times \ell^1$, the canonical action $\boldsymbol{\sigma}^\nu : GB^\infty_{+,k} \times (\ell^\infty \times \ell^1) \to \ell^\infty \times \ell^1$ defined in (4.24), and its equivariant momentum map $\mathcal{J}_\nu : \ell^\infty \times \ell^1 \to B^1_{-,k}$ given by (4.23) (see Proposition 4.1). By (3.7), the Banach Lie group $GB^\infty_{+,k}$ acts on the product $(\ell^\infty \times \ell^1) \times GL^\infty_{+,k} \times L^1_{-,k}$ by

$$h \cdot ((\mathbf{q}, \mathbf{p}), g, \rho) := \left(\boldsymbol{\sigma}^\nu(\mathbf{q}, \mathbf{p}), g \circ_k h^{-1}, (\mathrm{Ad}^{+,k})^*_{h^{-1}}\rho\right),$$

where $h \in GB^\infty_{+,k}$, $g \in GL^\infty_{+,k}$, $(\mathbf{q}, \mathbf{p}) \in \ell^\infty \times \ell^1$, and $\rho \in L^1_{-,k}$. This action admits the equivariant momentum map (3.8), which in this case becomes

$$\begin{aligned}&((\mathbf{q}, \mathbf{p}), g, \gamma + \gamma^\perp) \in (\ell^\infty \times \ell^1) \times GL^\infty_{+,k} \times \left(B^1_{-,k} \oplus (B^1_{-,k})^\perp\right)\\ &\quad \longmapsto \mathcal{J}_\nu(\mathbf{q}, \mathbf{p}) - \gamma \in B^1_{-,k}.\end{aligned}$$

The zero level set of this momentum map is a smooth manifold, $GB^\infty_{+,k}$-equivariantly diffeomorphic to $GL^\infty_{+,k} \times (\ell^\infty \times \ell^1) \times (B^1_{-,k})^\perp$, the action on the target being

$$h \cdot \left(g, \mathbf{q}, \mathbf{p}, \gamma^\perp\right) := \left(g \circ_k h^{-1}, \boldsymbol{\sigma}^\nu_h(\mathbf{q}, \mathbf{p}), \left(\mathrm{Ad}^{+,k}\right)^*_{h^{-1}} \gamma^\perp\right).$$

The symplectically induced space is hence the fiber bundle

$$GL^\infty_{+,k} \times_{GB^\infty_{+,k}} \left(\ell^\infty \times \ell^1 \times (B^1_{-,k})^\perp\right) \to GL^\infty_{+,k}/GB^\infty_{+,k}$$

associated to the principal bundle $GL^\infty_{+,k} \to GL^\infty_{+,k}/GB^\infty_{+,k}$.

We begin by explicitly determining the base manifold of this bundle. If $g = g_0 + \cdots + g_{k-1}S^{k-1} \in GL^\infty_{+,k}$ and $h = h_0 + h_{k-1}S^{k-1} \in GB^\infty_{+,k}$ then

$$\begin{aligned} g \circ_k h^{-1} &= (g_0 + \cdots + g_{k-1}S^{k-1}) \circ_k (h_0^{-1} - h_0^{-1}h_{k-1}s^{k-1}(h_0^{-1})S^{k-1})\\ &= g_0h_0^{-1} + g_1 s(h_0^{-1})S + \cdots + g_{k-2}s^{k-2}(h_0^{-1})S^{k-2}\\ &\quad + \left(g_{k-1}s^{k-1}(h_0^{-1}) - g_0h_0^{-1}h_{k-1}s^{k-1}(h_0^{-1})\right) S^{k-1}.\end{aligned}$$

Therefore, the smooth map $GL^{\infty}_{+,k} \to (\ell^{\infty})^{k-2}$ given by

$$\begin{aligned} GL^{\infty}_{+,k} \ni g_0 + \cdots + g_{k-1}S^{k-1} &\mapsto (g_0 + \cdots + g_{k-1}S^{k-1}) \circ_k (g_0^{-1} - g_0^{-1} g_{k-1} s^{k-1}(h_0^{-1}) S^{k-1}) \\ &= \mathbb{I} + g_1 s(g_0^{-1}) S + \cdots + g_{k-2} s^{k-2}(g_0^{-1}) S^{k-2} \\ &\mapsto \left(g_1 s(g_0^{-1}), \ldots, g_{k-2} s^{k-2}(g_0^{-1}) \right) \in \left(\ell^{\infty}\right)^{k-2} \end{aligned}$$

factors through the $GB^{\infty}_{+,k}$-action thus inducing a smooth map $GL^{\infty}_{+,k}/GB^{\infty}_{+,k} \to (\ell^{\infty})^{k-2}$. Its inverse is the smooth map

$$\left(\mathbf{q}_1, \ldots, \mathbf{q}_{k-2}\right) \in \left(\ell^{\infty}\right)^{k-2} \mapsto [\mathbb{I} + \mathbf{q}_1 S + \cdots + \mathbf{q}_{k-2} S^{k-2}] \in GL^{\infty}_{+,k}/GB^{\infty}_{+,k}$$

which proves that $GL^{\infty}_{+,k}/GB^{\infty}_{+,k}$ is diffeomorphic to $(\ell^{\infty})^{k-2}$.

Next, we shall prove that the smooth map

$$\Phi : (\ell^{\infty} \times \ell^1) \times \left(\ell^{\infty}\right)^{k-2} \times \left(\ell^1\right)^{k-2} \to GL^{\infty}_{+,k} \times_{GB^{\infty}_{+,k}} \left(\ell^{\infty} \times \ell^1 \times (B^1_{-,k})^{\perp}\right)$$

given by

$$\begin{aligned} &\Phi\left((\mathbf{q}, \mathbf{p}), \mathbf{q}_1, \ldots, \mathbf{q}_{k-2}, \mathbf{p}_1, \ldots, \mathbf{p}_{k-2}\right) \\ &\quad := \left[\left(\mathbb{I} + \mathbf{q}_1 S + \cdots + \mathbf{q}_{k-2} S^{k-2}, (\mathbf{q}, \mathbf{p}), S^T \mathbf{p}_1 + \cdots + (S^T)^{k-2} \mathbf{p}_{k-2}\right)\right] \end{aligned}$$

is a diffeomorphism thereby trivializing the associated bundle, which is the reduced space. Indeed, this map has a smooth inverse given by

$$\begin{aligned} &\Phi^{-1}\left(\left[\left(g_0 + \cdots + g_{k-1}S^{k-1}, (\mathbf{q}, \mathbf{p}), \gamma^{\perp}\right)\right]\right) \\ &\quad = \left(\boldsymbol{\sigma}^{\nu}_{g_0 + g_{k-1}S^{k-1}}(\mathbf{q}, \mathbf{p}), g_1 s(g_0^{-1}), \ldots, g_{k-2} s^{k-2}(g_0^{-1}), \left(\mathrm{Ad}^{+,k}\right)^*_{(g_0 + g_{k-1}S^{k-1})^{-1}} \gamma^{\perp}\right), \end{aligned}$$

where, in the third component of the right hand side we have identified $(B^1_{-,k})^{\perp}$ with $(\ell^1)^{k-2}$ through the isomorphisms $L^1_k \cong \ell^1$.

The $GL^{\infty}_{+,k}$-action on the reduced manifold $GL^{\infty}_{+,k} \times_{GB^{\infty}_{+,k}} \left(\ell^{\infty} \times \ell^1 \times (B^1_{-,k})^{\perp}\right)$ is given by

$$g' \cdot [g, (\mathbf{q}, \mathbf{p}), \gamma^{\perp}] = [g' \circ_k g, (\mathbf{q}, \mathbf{p}), \gamma^{\perp}]$$

for any $g', g \in GL^{\infty}_{+,k}$, $(\mathbf{q}, \mathbf{p}) \in \ell^{\infty} \times \ell^1$, and $\gamma^{\perp} \in (B^1_{-,k})^{\perp}$. Via the globally trivializing diffeomorphism Φ, the induced $GL^{\infty}_{+,k}$-action on $(\ell^{\infty} \times \ell^1) \times (\ell^{\infty})^{k-2} \times \left(\ell^1\right)^{k-2}$ has the expression

$$\begin{aligned} &(g_0 + \cdots + g_{k-1}S^{k-1}) \cdot \left((\mathbf{q}, \mathbf{p}), \mathbf{q}_1, \ldots, \mathbf{q}_{k-2}, \mathbf{p}_1, \ldots, \mathbf{p}_{k-2}\right) \\ &\quad = \Phi^{-1}\left((g_0 + \cdots + g_{k-1}S^{k-1}) \cdot \Phi\left((\mathbf{q}, \mathbf{p}), \mathbf{q}_1, \ldots, \mathbf{q}_{k-2}, \mathbf{p}_1, \ldots, \mathbf{p}_{k-2}\right)\right) \\ &\quad = \Phi^{-1}\left((g_0 + \cdots + g_{k-1}S^{k-1}) \cdot \left[\left(\mathbb{I} + \mathbf{q}_1 S + \cdots + \mathbf{q}_{k-2} S^{k-2}, (\mathbf{q}, \mathbf{p}), S^T \mathbf{p}_1 + \cdots + (S^T)^{k-2} \mathbf{p}_{k-2}\right)\right]\right) \\ &\quad = \Phi^{-1}\left(\left[\left((g_0 + \cdots + g_{k-1}S^{k-1}) \circ_k (\mathbb{I} + \mathbf{q}_1 S + \cdots + \mathbf{q}_{k-2} S^{k-2}), (\mathbf{q}, \mathbf{p}), S^T \mathbf{p}_1 + \cdots + (S^T)^{k-2} \mathbf{p}_{k-2}\right)\right]\right) \\ &\quad = \Phi^{-1}\left(\left[\left(g_0 + \sum_{l=1}^{k-2}\left(\sum_{i=0}^{l} g_{l-i} s^{l-i}(\mathbf{q}_i)\right) S^l + \left(\sum_{i=0}^{k-2} g_{k-1-i} s^{k-1-i}(\mathbf{q}_i)\right) S^{k-1}, (\mathbf{q}, \mathbf{p}),\right.\right.\right. \\ &\qquad \left.\left.\left. S^T \mathbf{p}_1 + \cdots + (S^T)^{k-2} \mathbf{p}_{k-2}\right)\right]\right) \end{aligned}$$

$$= \left(\boldsymbol{\sigma}^{\nu}_{g_0 + \left(\sum_{i=0}^{k-2} g_{k-1-i} s^{k-1-i}(\mathbf{q}_i) \right) S^{k-1}} (\mathbf{q}, \mathbf{p}), s(g_0^{-1}) \sum_{i=0}^{1} g_{1-i} s^{1-i}(\mathbf{q}_i), \ldots, s(g_0^{-1}) \sum_{i=0}^{k-2} g_{k-2-i} s^{k-2-i}(\mathbf{q}_i), \right.$$
$$\left. s(g_0) g_0^{-1}, \ldots, s^{k-2}(g_0) g_0^{-1} \right),$$

where the equality in the last $k-2$ components follows from (4.22).

Let us summarize the considerations above. Using (4.24) and denoting

$$\left((\mathbf{q}', \mathbf{p}'), \mathbf{q}_1', \ldots, \mathbf{q}_{k-2}', \mathbf{p}_1', \ldots, \mathbf{p}_{k-2}'\right) := (g_0 + \cdots + g_{k-1} S^{k-1}) \cdot \left((\mathbf{q}, \mathbf{p}), \mathbf{q}_1, \ldots, \mathbf{q}_{k-2}, \mathbf{p}_1, \ldots, \mathbf{p}_{k-2}\right),$$

we conclude that the $GL^{\infty}_{+,k}$-action on the reduced manifold $(\ell^{\infty} \times \ell^1) \times (\ell^{\infty})^{k-2} \times \left(\ell^1\right)^{k-2}$ is given by

$$\mathbf{q}' = \mathbf{q} + \log g_0 \tag{5.1}$$
$$\mathbf{p}' = \mathbf{p} + \left(\sum_{i=0}^{k-2} g_{k-1-i} s^{k-1-i}(\mathbf{q}_i) \right) g_0^{-1} \nu e^{s^{k-1}(\mathbf{q}) - \mathbf{q}} - \tilde{s}^{k-1} \left(\left(\sum_{i=0}^{k-2} g_{k-1-i} s^{k-1-i}(\mathbf{q}_i) \right) g_0^{-1} \nu e^{s^{k-1}(\mathbf{q}) - \mathbf{q}} \right) \tag{5.2}$$
$$\mathbf{q}_l' = s(g_0^{-1}) \sum_{i=0}^{l} g_{l-i} s^{l-i}(\mathbf{q}_i) \tag{5.3}$$
$$\mathbf{p}_l' = s^l(g_0) g_0^{-1} \mathbf{p}_l, \quad l = 1, \ldots, k-2. \tag{5.4}$$

All geometric objects described above satisfy the assumptions of Propositions 3.1 and 3.2 and thus one has the weak symplectic form Ω_k and the momentum map $\mathbf{J}_k : (\ell^{\infty} \times \ell^1) \times (\ell^{\infty})^{k-2} \times \left(\ell^1\right)^{k-2} \to L^1_{-,k}$ given by (3.9) and (3.11), respectively. By (4.11), $\mathbf{J}_k$ takes the form

$$\begin{aligned} &\mathbf{J}_k\left((\mathbf{q}, \mathbf{p}), \mathbf{q}_1, \ldots \mathbf{q}_{k-2}, \mathbf{p}_1, \ldots \mathbf{p}_{k-2}\right) \\ &= \left(\mathrm{Ad}^{+,k}\right)^*_{(\mathbb{I} + \mathbf{q}_1 S + \cdots + \mathbf{q}_{k-2} S^{k-2})^{-1}} \left(\mathcal{J}_{\nu}(\mathbf{q}, \mathbf{p}) + S^T \mathbf{p}_1 + \cdots + (S^T)^{k-2} \mathbf{p}_{k-2}\right) \\ &= \left(\mathrm{Ad}^{+,k}\right)^*_{(\mathbb{I} + \mathbf{q}_1 S + \cdots + \mathbf{q}_{k-2} S^{k-2})^{-1}} \left(\mathbf{p} + S^T \mathbf{p}_1 + \cdots + (S^T)^{k-2} \mathbf{p}_{k-2} + (S^T)^{k-1} \nu e^{s^{k-1}(\mathbf{q}) - \mathbf{q}}\right), \end{aligned} \tag{5.5}$$

where the inverse $(\mathbb{I} + \mathbf{q}_1 S + \cdots + \mathbf{q}_{k-2} S^{k-2})^{-1}$ is given by (4.7). We shall call $\mathbf{J}_k$ the *generalized Flaschka map*.

In order to obtain the explicit expression of the weak symplectic form Ω_k (see (5.10)) on the induced symplectic manifold $(\ell^{\infty} \times \ell^1) \times (\ell^{\infty})^{k-2} \times \left(\ell^1\right)^{k-2}$, let us notice that the symplectic form $\omega + \omega_L$ on $(\ell^{\infty} \times \ell^1) \times GL^{\infty}_{+,k} \times L^1_{-,k}$ is given by

$$\omega + \omega_L = -\mathbf{d}\left(\mathrm{Tr}(\mathbf{p} \mathbf{d} \mathbf{q}) + \mathrm{Tr}(\rho g^{-1} \circ_k \mathbf{d} g)\right), \tag{5.6}$$

where $g^{-1} \circ_k \mathbf{d} g$ is the left Maurer-Cartan form on the Banach Lie group $GL^{\infty}_{+,k}$. One has the following decomposition

$$\theta := \mathrm{Tr}(\rho g^{-1} \circ_k \mathbf{d} g) = \mathrm{Tr}\left(\sum_{l=0}^{k-1} \rho_l \theta_l\right) \tag{5.7}$$

for $\rho = \rho_0 + S^T \rho_1 + \cdots + (S^T)^{k-1} \rho_{k-1} \in L^1_{-,k}$ with

$$\theta_l = \sum_{i=0}^{l} h_i(g) s^i(\mathbf{d} g_{l-i}), \quad l = 0, 1, \ldots, k-1.$$

The diagonal operators h_i are the components of $g^{-1} = h_0 + h_1 S + \cdots + h_{k-1} S^{k-1}$ given by (4.7). Let $\tilde{\theta}$ be the pull back of θ to the zero level set of the momentum map (3.8). Next, we pull back the form $\tilde{\theta}$ to $(\ell^{\infty} \times \ell^1) \times (\ell^{\infty})^{k-2} \times \left(\ell^1\right)^{k-2}$

by the global section $\Sigma : (\ell^\infty \times \ell^1) \times (\ell^\infty)^{k-2} \times \left(\ell^1\right)^{k-2} \to GL^\infty_{+,k} \times \left(\ell^\infty \times \ell^1\right) \times (B^1_{-,k})^\perp$ defined by

$$\Sigma((\mathbf{q},\mathbf{p}),\mathbf{q}_1,\dots\mathbf{q}_{k-2},\mathbf{p}_1,\dots,\mathbf{p}_{k-2}) := \left(\mathbb{I}+\mathbf{q}_1 S, +\cdots+\mathbf{q}_{k-2}S^{k-2}, (\mathbf{q},\mathbf{p}), S^T\mathbf{p}_1+\cdots+(S^T)^{k-2}\mathbf{p}_{k-2}\right).$$

Therefore, we get

$$\begin{aligned}\Sigma^*\tilde{\theta} &:= \mathrm{Tr}(\mathbf{pdq}) + \mathrm{Tr}\left[(\mathcal{J}_\nu(\mathbf{q},\mathbf{p}))_0\,\theta_0\right] + \mathrm{Tr}\left((\mathcal{J}_\nu(\mathbf{q},\mathbf{p}))_{k-1}\,\theta_{k-1}\right) + \mathrm{Tr}\left(\sum_{l=1}^{k-2}\mathbf{p}_l\theta_l\right)\\ &= \mathrm{Tr}(\mathbf{pdq}) + \mathrm{Tr}\left(\sum_{l=1}^{k-2}\mathbf{p}_l\sum_{i=0}^{l-1}h_i(\mathbf{q}_1,\dots,\mathbf{q}_i)s^i(\mathbf{dq}_{l-i})\right)\\ &\quad + \mathrm{Tr}\left(\nu \mathrm{e}^{s^{k-1}(\mathbf{q})-\mathbf{q}}\sum_{i=1}^{k-2}h_i(\mathbf{q}_1,\dots,\mathbf{q}_i)s^i(\mathbf{dq}_{k-1-i})\right),\end{aligned} \tag{5.8}$$

since $\theta_0 = 0$, where $h_i(\mathbf{q}_1,\dots,\mathbf{q}_i)$ is given by (4.7) with $g_0 = (1,1,\dots)$, $g_1 = \mathbf{q}_1,\dots, g_{k-2} = \mathbf{q}_{k-2}$, $g_{k-1} = (0,0,\dots)$. Since $\mathrm{Tr}\,\delta = \mathrm{Tr}\,\tilde{s}^j(\delta)$ for any $\delta \in L^1_0$ and $j \in \mathbb{N}$, the last summand in (5.8) becomes

$$\begin{aligned}&\sum_{i=1}^{k-2}\mathrm{Tr}\left[\tilde{s}^i\left(\nu \mathrm{e}^{s^{k-1}(\mathbf{q})-\mathbf{q}}h_i(\mathbf{q}_1,\dots,\mathbf{q}_i)\right)\left(\mathbb{I}-\sum_{r=0}^{i-1}p_r\right)\mathbf{dq}_{k-1-i}\right]\\ &\quad = \sum_{i=1}^{k-2}\mathrm{Tr}\left[\tilde{s}^i\left(\nu e^{s^{k-1}(\mathbf{q})-\mathbf{q}}h_i(\mathbf{q}_1,\dots,\mathbf{q}_i)\right)\mathbf{dq}_{k-1-i}\right]\end{aligned}$$

because

$$\tilde{s}^j(\delta)\sum_{r=0}^{j-1}p_r = 0 \quad \text{for all } \delta \in L^1_0 \text{ and } j \in \mathbb{N}.$$

Similarly, the second summand in (5.8) equals

$$\sum_{l=1}^{k-2}\sum_{i=0}^{l-1}\mathrm{Tr}\left[\tilde{s}^i\left(\mathbf{p}_l h_i(\mathbf{q}_1,\dots,\mathbf{q}_i)\right)\mathbf{dq}_{l-i}\right],$$

so that (5.8) becomes

$$\begin{aligned}\Sigma^*\tilde{\theta} &= \mathrm{Tr}(\mathbf{pdq}) + \sum_{l=1}^{k-2}\mathrm{Tr}\left(\sum_{i=0}^{l-1}\tilde{s}^i\left(\mathbf{p}_l h_i(\mathbf{q}_1,\dots,\mathbf{q}_i)\right)\mathbf{dq}_{l-i} + \tilde{s}^l\left(\nu \mathrm{e}^{s^{k-1}(\mathbf{q})-\mathbf{q}}h_l(\mathbf{q}_1,\dots,\mathbf{q}_l)\right)\mathbf{dq}_{k-1-l}\right)\\ &= \mathrm{Tr}(\mathbf{pdq}) + \sum_{l=1}^{k-2}\left[\mathrm{Tr}\left(\sum_{i=0}^{k-2-l}\tilde{s}^i\left(\mathbf{p}_l h_i(\mathbf{q}_1,\dots,\mathbf{q}_i)\right) + \tilde{s}^l\left(\nu \mathrm{e}^{s^{k-1}(\mathbf{q})-\mathbf{q}}h_l(\mathbf{q}_1,\dots,\mathbf{q}_l)\right)\right)\mathbf{dq}_l\right].\end{aligned} \tag{5.9}$$

Then the reduced symplectic form is

$$\Omega_k = -\mathbf{d}\Sigma^*\tilde{\theta}. \tag{5.10}$$

Indeed, a straightforward verification shows that $-\mathbf{d}\Sigma^*\tilde{\theta}$ satisfies the condition characterizing the reduced symplectic form, so it must be equal to it. Note that the one-form $\Sigma^*\tilde{\theta}$ depends on the chosen section Σ, but that if $\tilde{\Sigma}$ is any other global section, then $\mathbf{d}\Sigma^*\tilde{\theta} = \mathbf{d}\tilde{\Sigma}^*\tilde{\theta} = \Omega_k$. In particular, the reduced symplectic form Ω_k is in this case exact. Note also that the symplectic form Ω_k is canonical only if $k = 2$ and magnetic only if $k = 3$, a case that we shall analyze in detail below. In general, if $k > 3$, the weak symplectic form Ω_k is neither canonical nor magnetic due to the presence of the $\mathbf{p}_j$-dependent coefficients of $\mathbf{dq}_l$ in the first sum of the second term.

Since $\mathbf{J}_k$ is a Poisson map and the functions I^k_l are in involution on L^1_k, it follows that $I^k_l \circ \mathbf{J}_k$ are also in involution on the weak symplectic manifold $\left((\ell^\infty \times \ell^1) \times (\ell^\infty)^{k-2} \times \left(\ell^1\right)^{k-2}, \Omega_k\right)$ provided that these functions admit Hamiltonian vector fields.

The case $k = 2$. Then we have $B^1_{-,2} = L^1_{-,2}$ and $GB^\infty_{+,2} = GL^\infty_{+,2}$. As we discussed earlier, the induction method yields, for this situation the original weak symplectic manifold $\left(\ell^\infty \times \ell^1, \omega\right)$. This is the case of the standard semi-infinite Toda lattice.

The case $k = 3$. This is the first situation that goes beyond the Toda lattice. The Banach Lie group $G := (GL^\infty_{+,3}, \circ_3)$ consists of bounded operators having only three upper diagonals, while the operators in $GB^\infty_{+,3}$ have non-zero entries only on the main and the second strictly upper diagonal. The induced space is now $(\ell^\infty \times \ell^1) \times (\ell^\infty \times \ell^1)$. The $GL^\infty_{+,3}$-action on $(\ell^\infty \times \ell^1) \times \left(\ell^\infty \times \ell^1\right)$ is given, according to (5.1)–(5.4) by

$$\mathbf{q}' = \mathbf{q} + \log g_0 \tag{5.11}$$

$$\mathbf{p}' = \mathbf{p} + g_2 g_0^{-1} \nu e^{s^2(\mathbf{q})-\mathbf{q}} + g_1 s(\mathbf{q}_1) g_0^{-1} \nu e^{s(\mathbf{q})-\mathbf{q}} - \tilde{s}^2 \left(g_2 g_0^{-1} \nu \mathrm{e}^{s^2(\mathbf{q})-\mathbf{q}} + g_1 s(\mathbf{q}_1) g_0^{-1} \nu \mathrm{e}^{s(\mathbf{q})-\mathbf{q}} \right) \tag{5.12}$$

$$\mathbf{q}_1' = s(g_0^{-1})(g_1 + g_0 \mathbf{q}_1) \tag{5.13}$$

$$\mathbf{p}_1' = s(g_0) g_0^{-1} \mathbf{p}_1, \quad l = 1, \ldots, k-2. \tag{5.14}$$

The reduced symplectic form on $(\ell^\infty \times \ell^1) \times (\ell^\infty \times \ell^1)$ is, according to (4.7), (5.9) and (5.10), equal to

$$\begin{aligned} \Omega_3 &= -\mathbf{d}\left[\operatorname{Tr}(\mathbf{p}\mathbf{d}\mathbf{q}) + \operatorname{Tr}\left(\mathbf{p}_1 \mathbf{d}\mathbf{q}_1\right) - \operatorname{Tr}\left(\nu e^{s^2(\mathbf{q})-\mathbf{q}} \mathbf{q}_1 s(\mathbf{d}\mathbf{q}_1)\right)\right] \\ &= -\mathbf{d}\left[\operatorname{Tr}(\mathbf{p}\mathbf{d}\mathbf{q}) + \operatorname{Tr}\left(\left(\mathbf{p}_1 - \tilde{s}\left(\nu e^{s^2(\mathbf{q})-\mathbf{q}} \mathbf{q}_1\right)\right)\mathbf{d}\mathbf{q}_1\right)\right] \\ &= -\mathbf{d}\left[\operatorname{Tr}(\mathbf{p}\mathbf{d}\mathbf{q}) + \operatorname{Tr}(\tilde{\mathbf{p}}_1 \mathbf{d}\mathbf{q}_1)\right], \end{aligned} \tag{5.15}$$

where

$$\tilde{\mathbf{p}}_1 := \mathbf{p}_1 - \tilde{s}\left(\nu e^{s^2(\mathbf{q})-\mathbf{q}} \mathbf{q}_1\right). \tag{5.16}$$

We see here exactly the same phenomenon as in classical electrodynamics, where a momentum shift by the magnetic potential transforms the non-canonical magnetic symplectic form to the canonical one.

The equivariant momentum map (5.5) of this action is by (4.11) and (5.16) equal to

$$\begin{aligned} \mathbf{J}_3\left(\mathbf{q}, \mathbf{p}, \mathbf{q}_1, \mathbf{p}_1\right) &= \left(\mathrm{Ad}^{+,3}\right)^*_{(\mathbb{I}+\mathbf{q}_1 S)^{-1}} \left(\mathbf{p} + S^T \mathbf{p}_1 + (S^T)^2 \nu e^{s^2(\mathbf{q})-\mathbf{q}}\right) \\ &= \mathbf{p} + \mathbf{q}_1 \mathbf{p}_1 - \tilde{s}\left(\mathbf{q}_1 \mathbf{p}_1 + s(\mathbf{q}_1) \nu e^{s^2(\mathbf{q})-\mathbf{q}} \mathbf{q}_1\right) + \tilde{s}^2\left(\nu e^{s^2(\mathbf{q})-\mathbf{q}} \mathbf{q}_1 s(\mathbf{q}_1)\right) \\ &\quad + S^T\left(\mathbf{p}_1 + s(\mathbf{q}_1)\nu e^{s^2(\mathbf{q})-\mathbf{q}} - \tilde{s}\left(\nu e^{s^2(\mathbf{q})-\mathbf{q}} \mathbf{q}_1\right)\right) + \left(S^T\right)^2 \nu e^{s^2(\mathbf{q})-\mathbf{q}} \\ &= \mathbf{p} + \mathbf{q}_1 \mathbf{p}_1 - \tilde{s}\left(\mathbf{q}_1 \mathbf{p}_1\right) - \tilde{s}\left(\nu e^{s^2(\mathbf{q})-\mathbf{q}} \mathbf{q}_1\right)\mathbf{q}_1 + \tilde{s}^2\left(\nu e^{s^2(\mathbf{q})-\mathbf{q}} \mathbf{q}_1\right)\tilde{s}(\mathbf{q}_1) \\ &\quad + S^T\left(\mathbf{p}_1 + s(\mathbf{q}_1)\nu e^{s^2(\mathbf{q})-\mathbf{q}} - \tilde{s}\left(\nu e^{s^2(\mathbf{q})-\mathbf{q}} \mathbf{q}_1\right)\right) + \left(S^T\right)^2 \nu e^{s^2(\mathbf{q})-\mathbf{q}} \\ &= \mathbf{p} + \mathbf{q}_1 \tilde{\mathbf{p}}_1 - \tilde{s}\left(\mathbf{q}_1 \tilde{\mathbf{p}}_1\right) + S^T\left(\tilde{\mathbf{p}}_1 + s(\mathbf{q}_1)\nu e^{s^2(\mathbf{q})-\mathbf{q}}\right) + \left(S^T\right)^2 \nu e^{s^2(\mathbf{q})-\mathbf{q}} \end{aligned} \tag{5.17}$$

since the inverse of $\mathbb{I} + \mathbf{q}_1 S$ in the Banach Lie group $GL^\infty_{+,3}$ is equal to $(\mathbb{I} + \mathbf{q}_1 S)^{-1} = \mathbb{I} - \mathbf{q}_1 S + \mathbf{q}_1 s(\mathbf{q}_1) S^2 \in GL^\infty_{+,3}$.

The Hamiltonians I^3_l are in involution on L^1_3 and hence the functions $I^3_l \circ \mathbf{J}_3$ are in involution on $\left((\ell^\infty \times \ell^1) \times \left(\ell^\infty \times \ell^1\right), \Omega_3\right)$, provided that they have Hamiltonian vector fields relative to the weak symplectic form Ω_3.

For $l = 1, 2$, the Hamiltonians $H_1 := I^3_1 \circ \mathbf{J}_3$ and $H_2 := I^3_2 \circ \mathbf{J}_3$ have the expressions

$$H_1(\mathbf{q}, \mathbf{p}, \mathbf{q}_1, \mathbf{p}_1) = \operatorname{Tr}(\mathbf{p}) \tag{5.18}$$

and

$$H_2(\mathbf{q}, \mathbf{p}, \mathbf{q}_1, \mathbf{p}_1) = \frac{1}{2} \operatorname{Tr}\left[\mathbf{p} + \mathbf{q}_1 \tilde{\mathbf{p}}_1 - \tilde{s}\left(\mathbf{q}_1 \tilde{\mathbf{p}}_1\right)\right]^2 + \operatorname{Tr}\left(\tilde{\mathbf{p}}_1 + s(\mathbf{q}_1)\nu e^{s^2(\mathbf{q})-\mathbf{q}}\right)^2 + \operatorname{Tr}\left(\nu e^{s^2(\mathbf{q})-\mathbf{q}}\right)^2. \tag{5.19}$$

The Hamiltonian system defined by H_2 describes a semi-infinite family of particles in an external field (given by the magnetic term of the symplectic form (5.15)) and where the interaction is between every second neighbor. In the case of the Toda lattice (obtained for $k = 2$, as discussed above), there is no external field and the interaction is between nearest neighbors. The semi-infinite Toda lattice is investigated in [8]. For arbitrary k there is an external field and the interaction of particles is between every $(k-1)$st neighbor.

Acknowledgments

A.O. thanks the Bernoulli Center for its hospitality and excellent working conditions during his extended stay there. We are grateful to D. Beltiţă and H. Flaschka for several useful discussions that influenced our presentation. The authors thank the Polish and Swiss National Science Foundations (Polish State Grant 1 P03A 001 29 and Swiss NSF Grant 200021-109111/1) for partial support.

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